A Technical Note: Derivation of the LRFD Column Design Equations

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INTRODUCTION

Since the publication of the first Load and Resistance Factor Design (LRFD) Specification by the American Institute of Steel Construction, Inc. (AISC, 1986) the column design equations have been presented by the Euler buckling equation in the elastic range and an exponential equation in the inelastic range. Background information, including accumulated test data, for these equations was presented at a Structural Stability Research Council (SSRC) meeting held in Cleveland, Ohio in 1985 (Tide, 1985). However, the actual derivation of the design equations relative to the accumulated test data has not been documented. The purpose of this paper is to document the derivation of the LRFD column design equations.

BACKGROUND

During the development of the LRFD Specifications in the early 1980s, various philosophical issues had to be resolved among the subcommittee members responsible for the derivation of the column design equations. The fundamental issues to be resolved were as follows:

- The large majority of structures are three stories tall or less. It is estimated that 75 to 90 percent of all steel columns erected in one year are contained in one-to-three story tall buildings (Tide, 1985). Column design for these types of buildings should not be more conservative than the current AISC Allowable Stress Design Specifications (ASD, 1978). This eliminated consideration of the SSRC #2 equations of the time (SSRC, 1976).
- The resulting column design equations, including the resistance factor (ϕ), should represent a lower bound to the test data.
- In the purely elastic range, the Euler buckling equation should be evident.
- The equations should be essentially continuous with no noticeable cusp at the common point. This meant that the equation coordinates and slopes should be very nearly equal at the common point.
- The reliability factor (beta) should come reasonably close to 3. Eventually, a local low beta of 2.6 was accepted (Tide, 1985).

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To satisfy all of the requirements, various forms were considered for the inelastic equations. The options considered included a parabolic equation, a polynomial equation and an exponential equation. The exponential equation appeared to be best at satisfying the previous criteria.

EQUATION DERIVATION

The derivation of the column equations will be shown in a non-dimensional format that is compatible with the AISC LRFD Specifications using the slenderness ratio (λ):

$$\lambda = \frac{KL}{\pi r}\sqrt{\frac{F_y}{E}} \quad (1)$$

where

K = effective length factor
L = column length, in.
r = radius of gyration, in.
F_y = yield stress of steel, ksi
E = modulus of elasticity of steel, ksi

To compare the ASD equations of the time (AISC, 1978) with proposed LRFD equations, an equivalent ASD safety factor (SF) and LRFD load factor (LF) is derived.

The equivalent SF (LF) can be expressed in terms of live load (LL) and dead load (DL) as follows:

$$\text{SF(LF)} = \frac{1.6\text{LL}+1.2\text{DL}}{\text{LL}+\text{DL}} \quad (2a)$$

Hereafter only the SF designation will be used. The right hand side numerator and denominator are divided by DL to give:

$$\text{SF} = \frac{1.6\left(\text{LL}/\text{DL}\right)+1.2}{\text{LL}/\text{DL}+1.0} \quad (2b)$$

Substituting LL/DL ratios of 0.35 and 3.0 into Equation 2b, and multiplying by 1.0/0.85 gives two values for the ASD equations converted to LRFD format. The resulting coefficients are 1.53 and 1.76 for LL/DL ratios of 0.35 and 3.0, respectively. Using the ASD (AISC, 1978) column equations in combinations with ϕ and SF coefficients, critical stresses can be computed for any value of *KL/r*.

The ASD equations, modified with two different values of the LL/DL and SF coefficients, and the SSRC #2 equation are plotted in Figure 1. As can be seen in this figure the SSRC #2 equation dips below the modified ASD equations.

This contradicts the first stipulation that was expressed at the time the LRFD Specification was developed.

Rearranging Equation 2b gives:

$$LL/DL = \frac{SF - 1.2}{1.6 - SF} \qquad (2c)$$

At $\lambda = 1.0$ ($KL/r = 89.2$ for A36 steel), the ASD allowable stress (AISC, 1978) is computed to be approximately 14.3 ksi. Arbitrarily, a point midway between the two ASD curves in Figure 1 indicates an F_{cr}/F_y ratio of approximately 0.56. A more precise value using the average of the two equations from the ASD LL/DL curves shown in Figure 1 is not warranted. This ratio indicates a nominal SF of 0.56 × 36/14.3 or 1.41. From Equation 2c, this gives a nominal LL/DL ratio of 1.1.

As indicated previously, the intent was to develop an exponential equation (3a) in the inelastic range and the Euler equation (3b) in the elastic range. These equations can be represented in non-dimensional form as:

$$F_1(\lambda) = F_{cr}/F_y = \phi\left[C_1^{\lambda^2}\right],\ \lambda \le 1.5 \qquad (3a)$$

$$F_2(\lambda) = F_{cr}/F_y = \phi\left[C_2\lambda^{-2}\right],\ \lambda \ge 1.5 \qquad (3b)$$

where λ^{-2} is the Euler equation.

Setting Equation 3a equal to the non-dimensional stress times the SF at $\lambda = 1.0$ allows a determination of the unknown coefficient C_1, as follows :

$$C_1^{\lambda^2} = \frac{1.41 \times 14.3}{36 \times 0.85} \qquad (3c)$$

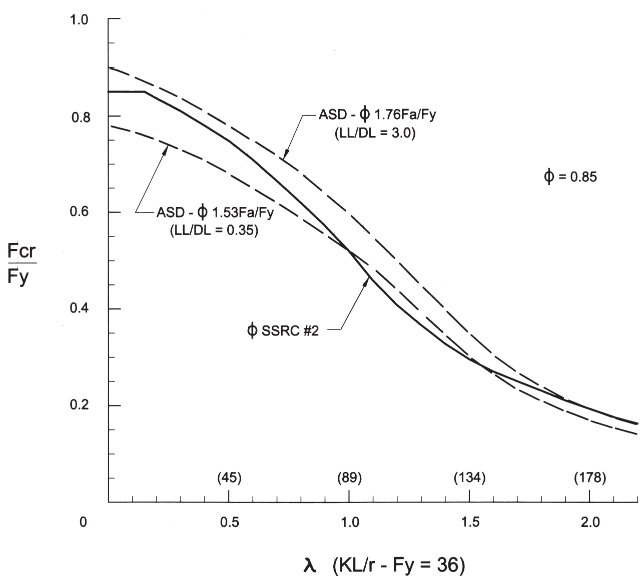

Fig. 1. Comparing SSRC #2 equation with LRFD-formatted ASD equations.

With $\lambda = 1.0$, $\lambda^2 = 1.0$ and $\phi = 0.85$, C_1 is computed to be equal to 0.6589 and the LRFD inelastic equation arbitrarily becomes:

$$F_{cr}/F_y = \phi\left[0.658^{\lambda^2}\right],\ \lambda \le 1.5 \qquad (4a)$$

Because EXP[X] = e^X, an alternative form of Equation 4a becomes:

$$F_{cr}/F_y = \phi\,\mathrm{EXP}\left[-0.419\lambda^2\right],\ \lambda \le 1.5 \qquad (4b)$$

Equations 3a (actually 4a) and 3b are equated at $\lambda = 1.5$ to determine the coefficient C_2.

$$\phi\left[C_2\lambda^{-2}\right] = \phi\left[0.658^{\lambda^2}\right]$$

from which $C_2 = 0.8774$.

The elastic Euler equation becomes:

$$F_{cr}/F_y = \phi\left[0.877\lambda^{-2}\right],\ \lambda \ge 1.5 \qquad (4c)$$

The common point at $\lambda = 1.5$ was chosen to approximate the historic ASD matching of elastic and inelastic curves at a stress level of $F_y/2$. For A36 steel, choosing an LRFD matching point slenderness $\lambda = 1.5$, results in an insignificant difference in KL/r: 126.1 for ASD and 133.7 for LRFD.

Comparing the results from the two Equations 4a and 4c at the common point of $\lambda = 1.5$ indicates non-dimensional critical stresses of 0.3315 and 0.3313, respectively. This is an insignificant difference of only 0.06 percent.

An overall comparison between the LRFD (AISC, 1986) equations and the comparable ASD (AISC, 1978) equations is shown in Figure 2.

EQUATION SLOPES AT MATCH POINT

To satisfy the requirement that no noticeable cusp occurs at the common point, the slopes of both equations were compared at $\lambda = 1.5$. Equations 4a and 4c are differentiated with respect to λ resulting in the slope equations:

$$\frac{d}{d\lambda}\left[F_1(\lambda)\right] = \phi\left[0.658^{\lambda^2}\left(\ln 0.658\right)2\lambda\right] \qquad (5a)$$

and

$$\frac{d}{d\lambda}\left[F_2(\lambda)\right] = \phi\left[0.877\lambda^{-3}\left(-2\right)\right] \qquad (5b)$$

With $\lambda = 1.5$, slopes of –0.416 and –0.442 are obtained from Equations 5a and 5b, respectively. Figure 3 is a graphical representation of the slopes of Equations 4a and 4c at the common point of $\lambda = 1.5$. The change in slope is so slight that it is hardly noticeable in Figure 3 and also insignificant in Figure 2.

SUMMARY

LRFD column equations were derived to represent a lower bound to the available test data and to satisfy the philosophical requirements of nearly all of the AISC Specification committee members at the time. Slight numerical differences occur at the common point for the two equations for both the non-dimensional stress and slope. These mismatches occur because an arbitrary condition was chosen to obtain the inelastic coefficient at $\lambda = 1.0$. The two equations were then forced to match at $\lambda = 1.5$.

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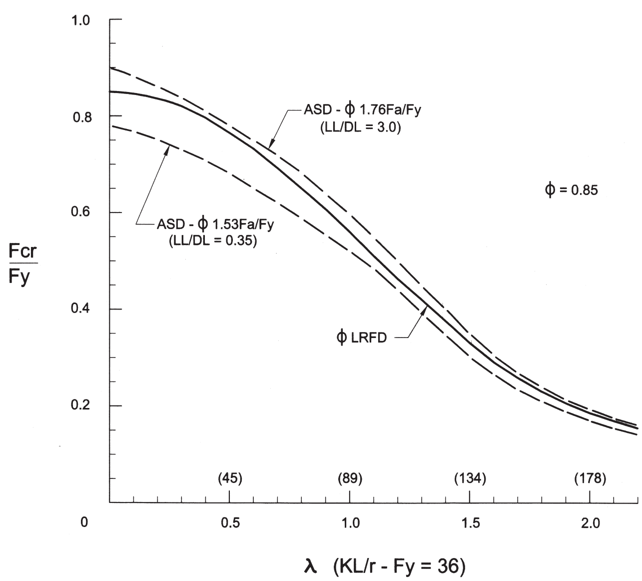

Fig. 2. Comparing AISC LRFD equations with LRFD-formatted ASD equations.

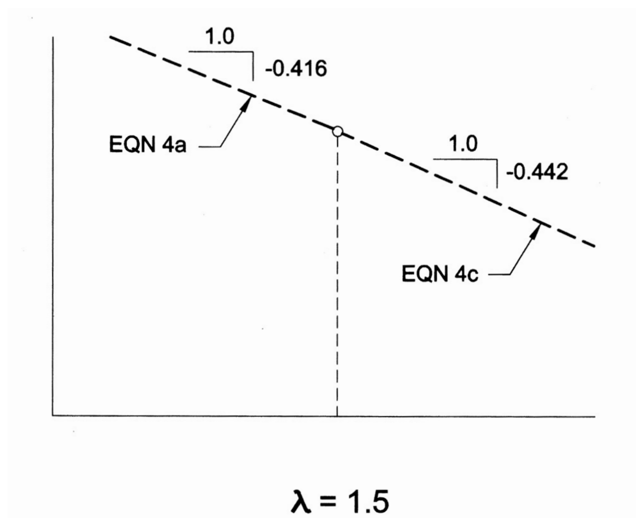

Fig. 3. Slopes of LRFD equations at common point.

DISCUSSION

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Discussion by LE-WU LU

The information presented by the author on the background and development of the LRFD column design equations is much appreciated. This discussion pertains to physical justifications of the equations and an issue related to the classroom teaching of column design. A common practice in teaching the design of a particular type of structural member (tension, compression, flexural, etc), is to first describe, qualitatively, the behavior and limit states of the member, then proceed to develop analytical predictions using basic mechanics of materials principles. The predictions, together with the available experimental and/or numerical results are then used to explain the applicable design provisions, as they are approximations or simplifications to the analytical predictions. This process provides the students with a good appreciation of the provisions. This however, is not the case for column design using the LRFD equations.

The column equations were developed as a reasonable lower bound to over 300 column test results (Tide, 1985) and provide column strength predictions close to the probability-based SSRC column curve 2P (Bjorhovde, 1972; SSRC, 1976; Iwankiw, 1985). Conceptually, they do not represent the strength of any particular column. However, studies have shown that it does provide good predictions of the strength of an initially crooked, as rolled, A36 steel W8×31 column having a small end restraint and failing by flexural instability about its minor axis. A comparison of the LRFD equations with the analytically calculated curve for this column is given in Figure 1. The curve was developed using a numerical procedure presented by Shen and Lu (1983). The calculations were based on the Lehigh type linear residual stress distribution with a maximum flange tip compressive stress of 0.3 F_y. The initial crookedness was assumed to be a sine curve with a maximum value v_m = L/1500, where L is the column length. This value is close to the statistical mean of the measured crookednesses (Bjorhovde, 1972). Equal end restraint with a rotational stiffness equal to 0.1*EI/L*, where *EI* is the flexural stiffness of the column about its minor axis, was assumed. The curve was calculated directly for the initially crooked column with the assumed end restraint and no use was made of any effective length factor, *K*, which is valid only for initially straight columns. However, as a convenient reference, the *K* value of this column (if it is initially straight and buckles elastically) is 0.98. Other wide flange columns having similar yield stress, initial crookedness and end restraint have approximately the same predicted strengths. This curve also gives conservative estimates of the strengths of a variety of other columns (columns failing by instability about the major axis, high strength steel columns, box columns, etc.).

In the classrooms, this information can be presented to the students before explaining the probabilistic nature of

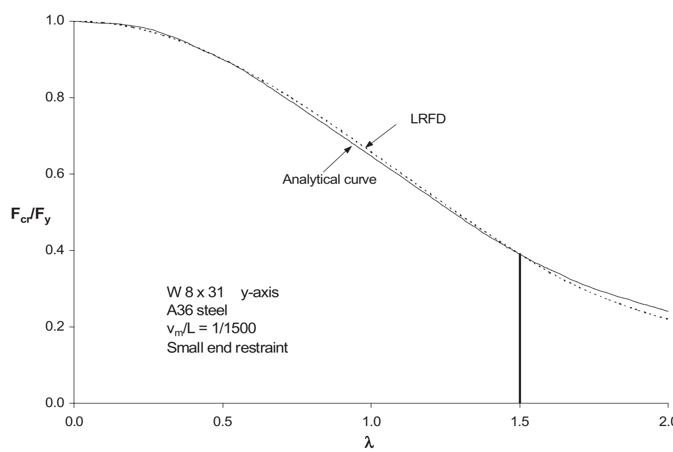

Fig. 1. Comparison of column strength curves.

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column strength and the LRFD equations. This helps the students develop a better understanding and appreciation of the design provisions.

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