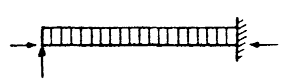
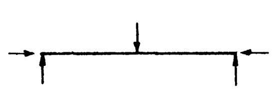
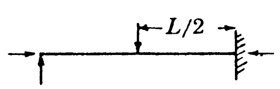
Note on Beam-Column Moment Amplification Factor

NESTOR R. IWANKIW

This short note has been prepared to clarify and explain the theoretical derivation of the expressions shown in Table C1.6.1 of the AISC Specification Commentary.

Table C1.6.1.

Case	ψ	C_m
	0	1.0
	−0.4	$1 - 0.4\frac{f_a}{F'_e}$
	−0.4	$1 - 0.4\frac{f_a}{F'_e}$
	−0.2	$1 - 0.2\frac{f_a}{F'_e}$
	−0.3	$1 - 0.3\frac{f_a}{F'_e}$
	−0.2	$1 - 0.2\frac{f_a}{F'_e}$

This table shows suggested bending moment amplification factor coefficients, ψ and C_m, for braced transversely loaded beam-columns which are to be used in conjunction with the Specification stability interaction Formula 1.6-1a. This C_m moment reduction factor may be written as

$$C_m = 1 + \psi \frac{f_a}{F_e'} \tag{1}$$

where

f_a = computed axial stress
F_e' = Euler buckling stress, divided by factor of safety

As the Commentary states, ψ for single curvature *simply supported* members may be approximated by

$$\psi = \frac{\pi^2 \delta_o EI}{M_o L^2} \tag{2}$$

where

δ_o = maximum deflection due to transverse loading
M_o = maximum moment between supports due to transverse loading

In the previous 7th Edition of the AISC Manual, Eq. 2 was erroneously used to also determine ψ values for the four indeterminate end restrained cases.

Prof. J. A. Yura, University of Texas at Austin, calculated the correct values for amplified moments on the basis of exact theoretical solutions presented by Timoshenko and Gere. He then developed approximate expressions for ψ and C_m that are contained in the current Manual edition. A brief summary of this derivation is presented in Table 1 (see next page). The two simply supported cases shown in Commentary Table C1.6.1 were not affected by the changes.

It is hoped this will resolve some of the common inquiries relative to ψ and Table C1.6.1.

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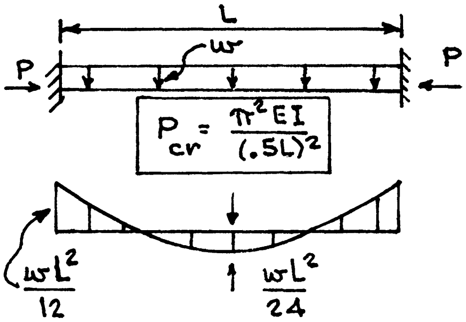
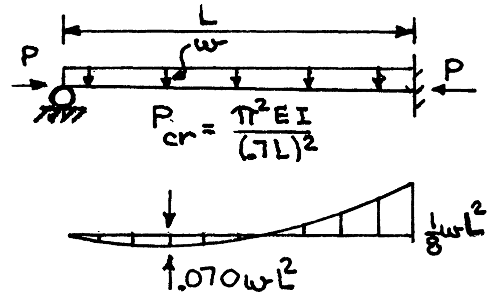
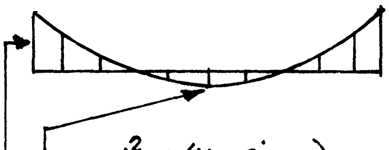
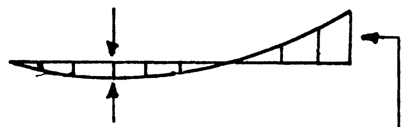
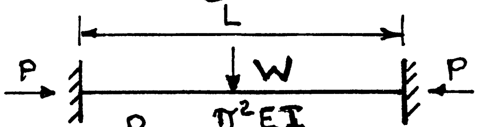
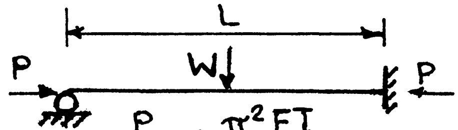
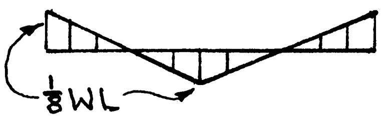
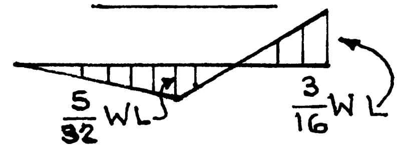
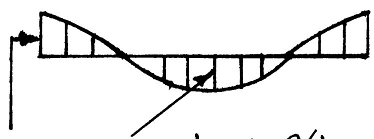
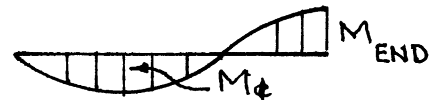
Table 1. Derivation of ψ Factors for End-Restrained Members

CASE	①	②
		
M Diagram without P	$P_{cr} = \frac{\pi^2 EI}{(.5L)^2}$; end moment $\frac{wL^2}{12}$, centerline moment $\frac{wL^2}{24}$	$P_{cr} = \frac{\pi^2 EI}{(.7L)^2}$; end moment $\frac{1}{8}wL^2$, span moment $.070wL^2$
Exact Solution with P (1)		
$u = \frac{L}{2}\sqrt{\frac{P}{EI}}$ OR $u = \frac{\pi}{2K}\sqrt{\alpha}$	$M_{℄} = \frac{wL^2}{24}\,\frac{6(u - \sin u)}{u^2 \sin u}$ $M_{END} = \frac{wL^2}{12}\,\frac{3(\tan u - u)}{u^2 \tan u}$	$M_{℄}$ - not calculated, obvious it cannot control $M_{END} = \frac{wL^2}{8}\,\frac{(\tan u - u)}{\frac{1}{2}u^2\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)}$

	Amplified Moments × $wL^2/24$			Amplified Moments × $wL^2/8$		
$\alpha = P/P_{cr}$	$M_{℄}$	M^*_{END}	M_{END} (APPROX.)	$M_{℄}$	M^*_{END}	M_{END} (APPROX.)
0.0	1.000	2.000	2.00	Does not control	1.000	1.00
0.1	1.129	2.146	2.14		1.074	1.07
0.2	1.291	2.324	2.30		1.166	1.15
0.3	1.501	2.554	2.52		1.282	1.26
0.4	1.782	2.854	2.80		1.434	1.40
0.5	2.179	3.272	3.22		1.645	1.61
0.6	2.778	3.892	3.83		1.959	1.92
0.7	3.782	4.922	4.85		2.475	2.42
0.8	5.798	6.960	6.88		3.493	3.44
0.9	11.87	13.08	12.98		6.481	6.49
	*always max.			*always max.		

	①	②
Approximate Solution	$M_{END} = \frac{wL^2}{12}\left(\frac{1 - 0.39\,P/P_{cr}}{1 - P/P_{cr}}\right)$ K = 0.5 ∴ Ψ = −0.4	$M_{END} = \frac{wL^2}{8}\left(\frac{1 - 0.39\,P/P_{cr}}{1 - P/P_{cr}}\right)$ K = 0.7 ∴ Ψ = −0.4

(1) Timoshenko & Gere, Theory of Elastic Stability, Chapter 1

CASE	③	④
	 $P_{cr} = \frac{\pi^2 EI}{(.5L)^2}$	 $P_{cr} = \frac{\pi^2 EI}{(.7L)^2}$
M DIAGRAM without P	 $\frac{1}{8}WL$	 $\frac{5}{32}WL$, $\frac{3}{16}WL$
EXACT SOLUTION with P: $\mu = \frac{L}{2}\sqrt{\frac{P}{EI}}$ OR $\mu = \frac{\pi}{2K}\sqrt{\alpha}$, $\alpha = P/P_{cr}$	 $M_{END} = M_{℄} = \frac{1}{8}WL\,\frac{2(1-\cos\mu)}{\mu\sin\mu}$	 $M_{℄} = \frac{5WL}{32}\left[1.6\,\frac{\tan\mu}{\mu} - 0.6\sec\mu\,\frac{\lambda(\mu)}{\psi(\mu)}\right]$ $M_{END} = \frac{3}{16}WL\,\frac{\lambda(\mu)}{\psi(\mu)}$ WHERE: $\lambda(\mu) = 2(1-\cos\mu)/\mu^2\cos\mu$ $\psi(\mu) = \frac{3}{2\mu}\left(\frac{1}{2\mu} - \frac{1}{\tan 2\mu}\right)$

Amplified Moments × WL/8 (Case ③); Amplified Moments × $\frac{3}{16}$ WL (Case ④)

α	③ $M_{℄}$	③ M^{*}_{END}	③ M_{END} (APPROX.)	④ $M_{℄}$	④ M^{*}_{END}	④ M_{END} (APPROX.)
0.0	$M_{℄} = M_{END}$	1.000	1.00	.833	1.00	1.00
0.1		1.091	1.09	.900	1.083	1.08
0.2		1.205	1.20	.982	1.186	1.17
0.3		1.351	1.35	1.086	1.317	1.30
0.4		1.543	1.54	1.222	1.488	1.47
0.5		1.817	1.81	1.412	1.725	1.70
0.6		2.224	2.21	1.908	2.076	2.05
0.7		2.901	2.89	2.152	2.653	2.63
0.8		4.254	4.24	3.059	3.790	3.80
0.9		8.308	8.29	5.722	7.122	7.30

Case ③: *equal to $M_{℄}$

$$M_{END} = M_{℄} = \frac{WL}{8}\left(\frac{1 - 0.19\,P/P_{cr}}{1 - P/P_{cr}}\right)$$

$K = 0.5$

$\therefore \Psi = -0.2$

Case ④: *always max.

$$M_{END} = \frac{3}{16}WL\left(\frac{1 - 0.30\,P/P_{cr}}{1 - P/P_{cr}}\right)$$

$K = 0.7$

$\therefore \Psi = -0.3$

DISCUSSION

Note on Beam-Column Moment Amplication Factor

Paper by NESTOR IWANKIW
(1st Quarter, 1984)

Discussion by **Le-Wu Lu**

The author has presented the background material that has led to changes in some of the C_m factors given in Table C1.6.1 of the AISC Specification Commentary. The changes are for propped cantilevers and fixed-end members subjected to combined axial compression and lateral load. Whether the new factors are more appropriate than the factors given in the previous editions of the Commentary depends on the interpretation of the "spirit" of the beam-column interaction formula. Formula 1.6-1a (for the case of uniaxial bending) is essentially an allowable-stress format presentation of Formula 2.4-2, which has been developed semi-empirically from the ultimate strength solutions of beam-columns under axial compression P and equal and opposite bending moments M applied at the ends. Empirical C_m factors are then used for other cases of applied moments. Formula 2.4-2 together with the recommended C_m factors provides estimates of the combinations of P and M which would cause individual beam-columns to fail by instability. The spirit of the formula and Formula 1.6-1a is therefore to deal with the ultimate state of individual members which make up a complete structure.

If the six cases of laterally loaded columns of Table C1.6-1 are treated as individual members, the appropriate interaction formulas should define the limit combinations of axial compression and lateral load which would produce failure. Formula 2.4-2 should therefore be written in terms of P and the distributed lateral load w (for the first three cases) or the concentrated load R (for the last three cases). The appropriate interaction formulas are

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$$\frac{P}{P_{cr}} + \frac{C_m w}{(1 - P/P_e)\, w_p} \leq 1.0 \qquad (1)$$

and

$$\frac{P}{P_{cr}} + \frac{C_m R}{(1 - P/P_e)\, R_p} \leq 1.0 \qquad (2)$$

In these equations, w_p and R_p are the plastic limit loads and all the other terms are as defined in the Specification. In a paper by the writer and H. Kamalvand (Ref. 1), ultimate strength solutions for four of the six cases have been presented and the C_m factors given in the previous editions of the Commentary have been found to be quite satisfactory. These factors are:

$C_m = 21.0$	for simply supported column subjected to uniformly distributed load
$C_m = 1 - 0.4 \dfrac{P}{P_e}$	for fixed end column subjected to uniformly distributed load
$C_m = 1 - 0.2 \dfrac{P}{P_e}$	for simply supported column subjected to a concentrated load at mid-span
$C_m = 1 - 0.6 \dfrac{P}{P_e}$	for fixed end column subjected to a concentrated load at mid-span

Equations 1 and 2 and the C_m values given above should again be considered as empirical formulas and their validity has been established by comparison with analytically obtained ultimate strength solutions. These equations may be converted to the form of Formula 2.4-2, which expresses the interaction between P and M. The required conversion is straightforward for the simply supported column because the maximum moment occurring at the mid-span is directly related to the applied load. For the fixed end column, the analytical studies reported in Ref. 1 indicate that yielding often occurs at both the fixed ends and the mid-span, when the limit combination of P and w (or R) is reached. This situation is similar to that of a fixed-end beam when the applied lateral load reaches the plastic limit load. This observation allows an approximate conversion of w or R to M. For the case of uniform load, $w_p = 16M_p/L^2$, in which M_p is the plastic moment of column cross section and L the length. Similarly, the lateral load w can be expressed in terms of an equivalent moment M and L, $w = 16M/L^2$. Equation 2 can therefore be written as:

$$\frac{P}{P_{cr}} + \frac{C_m M}{(1 - P/P_e) M_p} \leq 1.0 \qquad (3)$$

in which $M = wL^2/16$. Similarly, for the case of concentrated load, $M = RL/8$.

The new C_m factors described by the author are approximations based on the theoretical solutions of elastic columns. If the spirit of the beam-column interaction formula is to deal with the ultimate state, it would be inappropriate to use C_m factors derived from elastic considerations (unless they happen to work out). Furthermore, the information given in the paper indicates that the maximum elastic moment due to lateral load is to be used in calculating f_b. This is unnecessarily conservative.

Refer again to the fixed end column. According to the recommended procedure, the term $C_m f_b$ in Formula 1.6-1a is given by

$$C_m f_b = (1 - 0.4 \frac{f_a}{F'_e}) \frac{wL^2}{12S} \qquad (4)$$

for the uniform load case, and

$$C_m f_b = (1 - 0.2 \frac{f_a}{F'_e}) \frac{RL}{8S} \qquad (5)$$

for the concentrated load case, in which S is the section modulus and f_a and f'_e are as defined in the Specification. The procedure based on the ultimate strength concept, as explained above, uses the following expressions for $C_m f_b$:

$$C_m f_b = (1 - 0.4 \frac{f_a}{F'_e}) \frac{wL^2}{16S} \qquad (6)$$

and

$$C_m f_b = (1 - 0.6 \frac{f_a}{F'_e}) \frac{RL}{8S} \qquad (7)$$

The values of $C_m f_b$ calculated by Eqs. 6 and 7 are always smaller than those by Eqs. 4 and 5 and may therefore provide more economical designs for laterally loaded columns. Equations 6 and 7 are of course approximate and empirical expressions but are consistent with the basic concept of the beam-column interaction formulas.

REFERENCE

1. *Lu, L. W. and H. Kamalvand* Ultimate Strength of Laterally Loaded Columns *Journal of the Structural Division, ASCE, Vol. 94, No. ST1, June, 1968.*

Closing discussion by **Joseph A. Yura**

I have been asked to respond to the discussion by Lu, since I proposed the C_m values for laterally loaded beam-columns in the current AISC Specification.

Lu raises two main issues. First, the "spirit" of the AISC Formula 1.6-1a should give the ultimate strength of a member in a frame, that is, redistribution of moments should be considered. Second, that C_m is an empirical factor used to equate the interaction equation with an exact theoretical solution.

The AISC Specification Advisory Committee has not adopted redistribution of moments directly in the interaction equation (after much discussion) because of the broad range of problems Formula 1.6-1a is expected to handle satisfactorily. Non-compact sections, laterally supported beam-columns and sections such as angles used in the top chords of trusses and joists are examples of situations in which redistribution is not appropriate. The theoretical solutions developed in Lu's Ref. 1 are limited to strong-axis bending of W shapes with L/r no greater than 100. These solutions are too limited for use in Part 1 of the AISC Specification. They may be appropriate for use in Part 2 — Plastic Design, which will be discussed later.

All values of C_m published in the AISC Specification since the adoption of Formula 1.6-1a in 1963 are used to estimate the *elastic* second order moment. This same philosophy is used in the new LRFD specification. In LRFD the moment used in the interaction equation is based on an elastic second order analysis. The engineer can choose to perform a second order analysis, in which case C_m would not even be used. C_m is only used when

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a first order structural analysis is performed. C_m is not material related; the same amplification factors are used in reinforced concrete and wood codes. Unfortunately, because of mathematical errors in the derivation of C_m values which appeared in the 1963 AISC Specification, these same errors appear in other codes.

Lu states that, for beams with fixed ends, the values of C_m in the current AISC Specification are unnecessarily conservative based on solutions he developed in Ref. 1. Unfortunately, Lu's ultimate strength solutions are incorrect. Similar mathematical mistakes that were made in the development of the 1963 C_m factors (1-0.4 f_a/F'_e), for a midspan moment of $wl^2/24$ for uniform load and (1-0.6 f_a/F'_e) for concentrated load, were made in Ref. 1. Both solutions assume that in the elastic range the end moments are not changed by second order effects as shown in Figs. 1a-1d. For the case of a concentrated load at midspan Lu states, "An elementary analysis will show that the bending moment is generally higher at the midspan than at the ends." Timoshenko's analysis cited in the article shows that the end moment and midspan moment are identical for all values of axial load and slenderness ratio. Also, area-moment principles require that the area of the moment diagram (*EI* being constant) between midspan and one end add up to zero since the slopes of the deflected shape in the elastic range are zero at these two locations. The moment diagram in Fig. 1d does not satisfy the boundary conditions. The correct diagram is shown in Fig. 1e. Since the end moment and midspan moment are the same, plastic hinges will form at the midspan and ends simultaneously. The ultimate strength of the fixed end beam with a concentrated load at midspan must be the same as a simple beam with one-half the span since there are inflection points at the quarter points in both the elastic and plastic ranges. Lu's solution for the simply supported beam case is correct and C_m = 1-0.2f_a/F'_e was found to be satisfactory. This same value of C_m must be correct for the fixed end beam, also. The values of C_m in the latest AISC Specification use this value of C_m for both simple and fixed end beams. In Lu's analysis a boundary condition of end slope always equal to zero was used in the inelastic range, which is incorrect.

For the uniformly loaded beam, where the elastic end moment is twice the midspan moment, Lu states, "In columns with high slenderness. . . .the center moment may become larger than the moment at the two ends, and initial yielding is likely to occur at midspan." A correct elastic second order analysis as given in the article shows that the maximum moment, hence first yielding, will always occur at the fixed end, not midspan, regardless of the magnitude of column axial load or slenderness ratio.

In summary, the new C_m values are accurate and not unduly conservative. The redistribution philosophy suggested by Lu may be appropriate for C_m values in Part 2 of the AISC Specification. However, there is a reduced potential saving for the uniformly loaded fixed end beam. With zero axial load, the ratio of midspan moment to fixed end moment is 0.5, thus permitting an average moment of $wL^2/16$ for plastic design. When an axial load of $0.6P_{cr}$ is applied, the ratio of elastic second order moments increases to 0.72, indicating a reduced potential redistribution of moments. The ratio of elastic second-order moments approaches 1.0 as the axial load approaches P_{cr}.

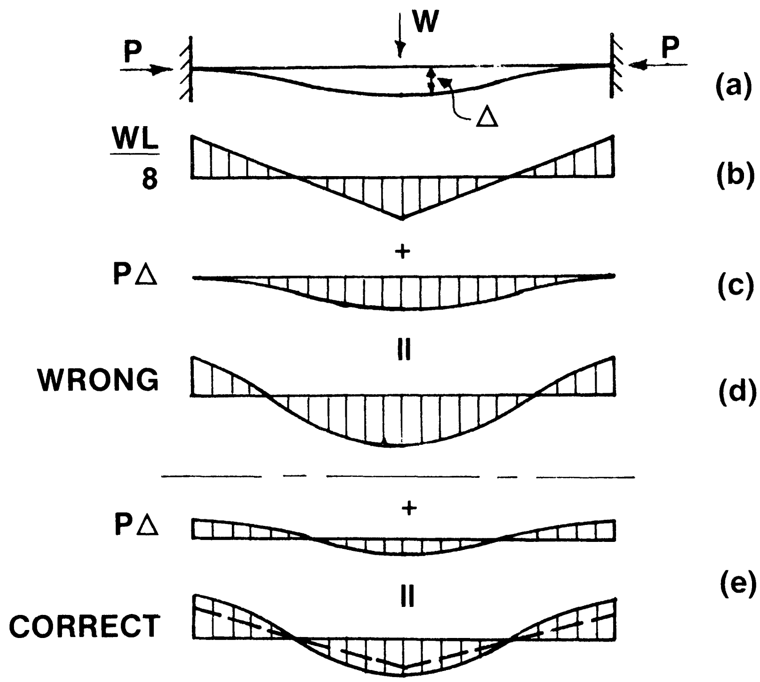

Fig. 1. Second order elastic moment diagrams