Effect of End Restraint on Column Strength—Practical Applications

REIDAR BJORHOVDE

INTRODUCTION

The behavior, strength and design of columns constitute a subject area that probably has received more study and discussion than most structural engineering problems. As a result, a large number of increasingly more accurate solutions have been developed, and the design approaches are legion.[1]

The basic column is a centrally loaded, pinned-end member. The original solution for the elastic case was provided by Euler in 1759, and advances over the years have incorporated the influences of residual stresses, initial crookedness and load eccentricity, to mention some of the primary factors.[1,2] Recognizing that columns are rarely, if ever, pinned-end in actual structures, but rather connected to the other components of the structure by a variety of connection types, recent studies have developed solutions that also take into account the effect of end restraint.[3,4,5,6] As can be expected, there is good agreement between theoretically determined column strengths and those obtained in column tests. This has been facilitated by the use of computer programs that can take into account the many column strength parameters,[2,4,6] including their random characteristics.[2]

The basic pinned-end column has traditionally been used as the anchor point or reference member with which real columns are compared and designed. This has been a convenient design tool, because of the extent of the knowledge of the behavior and strength of the pinned-end member. However, it has been recognized all along that differences in end support conditions, particularly as regards columns in frames, have significant effects on the response of the members. The strength-increasing influence of rigid supports, for example, and the strength-reducing influence of column sway are well documented and understood.[1] This led to the development of the concept of effective length, whereby the strength of a real column with realistic end support conditions could be determined from that of an equivalent pinned-end one. The length of the latter was equal to the effective length of the former, and elastic as well as inelastic solutions for the stability of the perfectly straight column showed that the distance between the inflection points of the buckled member was equal to KL, the effective length. In other words, for clearly defined end conditions it is a relatively easy problem to determine the column strength. Recent research has also shown that the concept may be applied to initially curved columns.[3]

The solution of the column problem is significantly more complicated when the member is part of a structural frame. The sizes of the members that are framed, the types of beam-to-column connections, and whether the frame is sway-permitted or sway-prevented play important roles insofar as the column strength is concerned. The subassemblage stability solution that was provided by the Julian and Lawrence nomographs for the effective length factors in frames[7] was admittedly flawed because of the many assumptions[8] that were used to arrive at a practicable design approach. However, it gave a rational and reasonably realistic solution to a difficult problem, and has therefore been used extensively by designers.

Current frame analysis methods may eventually lead to the dissappearance of the use of the K-factor in design, because large-scale computer programs facilitate the evaluation of the overall stability of the frame. However, it is also clear that many structures and design approaches will continue to be based on the performance of the individual structural members. For example, many low-rise buildings will undoubtedly be analyzed as they have been in the past. For sway-prevented structures (braced frames) of this kind, current designs call for effective length factors of 1.0, using a conservative simplification for the column and its support conditions. It is the purpose of this paper to evaluate the behavior and strength of columns in such structures, with the view towards developing more economical and realistic design criteria and procedures.

Reidar Bjorhovde is Professor of Civil Engineering and Engineering Mechanics, University of Arizona, Tucson, Arizona. The paper was presented at AISC's 1983 National Engineering Conference in Memphis, Tennessee.

SCOPE OF STUDY

The study presented in the following deals only with the behavior and strength of columns in sway-prevented structures, for which a common design approach utilizes a K-factor of 1.0. The evaluation will consider frames that are normally designated as simply connected, defined as Type 2 construction under the AISC *Specification*.[9] The beam-to-column connections that are used in such frames are regarded as simple, having no moment resistance, and are therefore treated as non-contributing to the lateral stability of the frame. By the same token, these connections therefore are assumed to offer no restraint to the buckling of the columns. Double angles, single-plate shear tabs, and top-and-seat angle connections are representative of this kind; they offer some moment capacity, but only on the order of 5 to 20% of that of a rigid connection.[10]

The study also deals with the influence of the beam stiffness on the stability of the column, considering it is connected to the latter by simple connections. In addition, the paper examines the column thus restrained as part of a frame; specifically, the method of determination of K-factors for restrained columns will be developed. This also requires an understanding of the buckling behavior of the column, including the influence of whether the member is connected to one or two beams at the ends. In other words, it is necessary to examine whether the column is an exterior or an interior member. Finally, design procedure recommendations will be presented.

THE PROBLEM AND ITS BACKGROUND

A typical sway-prevented frame and its assumed pinned beam-to-column connections is shown in Fig. 1, along with an indication of the buckling shape of the column in the structure. The service core shown is, of course, but one of many ways the structure can be braced to achieve the necessary lateral force resisting system. The beams are assumed to be pinned at the ends, and are therefore designed as simply supported members. The columns are assumed to be continuous, and because of the sway-preventing action of the core, the buckled shape of the column in each story closely resembles that of a pinned-end member. As a result, common design practice features a column design that is based on an effective length factor of K = 1.0, and there is no need to make use of an alignment chart such as is used for a sway-permitted frame. However, it is noted that the Julian and Lawrence[7] stability solution included a nomograph for the sway-prevented structure, although the beam-to-column connections of the frame in Fig. 1 would make it unfit for analysis by that alignment chart. In other words, the nomograph was developed on the basis of rigid (i.e. moment-resistant) connections.

As a result of the above design approaches, the sway-prevented nomograph for column design is normally not even published in many design manuals, although the column design section of the AISC Manual[11] incorporates it along with that for the sway-permitted case.

The types of actual beam-to-column connections that are regarded as pins are the double angle, the single angle, the single-plate shear tab, the top-and-seat angle, and the header plate connections. The most commonly used American types are shown schematically in Fig. 2. In actual fact, the assumption that these connections behave as pins is not correct, since they may have ultimate moment capacities between 5% and 20% of the rigid ones, depending on the size and type of connection. This is demonstrated by the moment-rotation (M-ϕ) curves shown in Fig. 3.

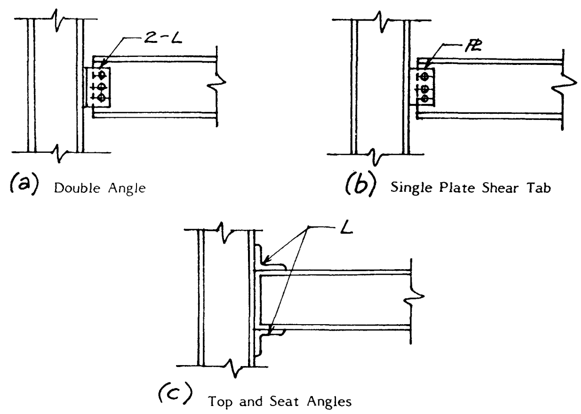

Fig. 2. Schematic illustration of most common American simple beam-to-column connection

More important than the ultimate moment capacity, however, is the initial stiffness of the connections. This is represented by the slope of M-ϕ-curve at the origin, designated as the connection rotational stiffness, C. For a perfect pin the rotational stiffness is zero; the actual C-values for some typical simple connections are given in Table 1,[12] and additional data and approaches for finding C have been provided by other studies.[6,10,13]

On the basis of the above, the premise is: Given that actual connections in Type 2 construction do offer a certain

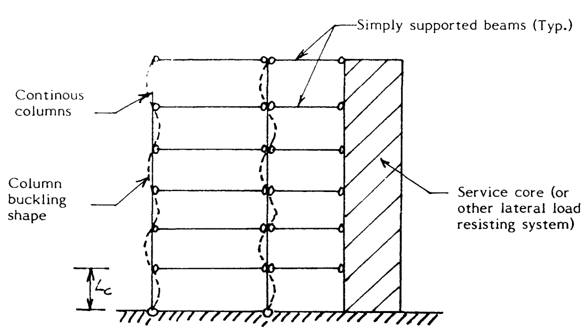

Fig. 1. Typical sway-prevented frame

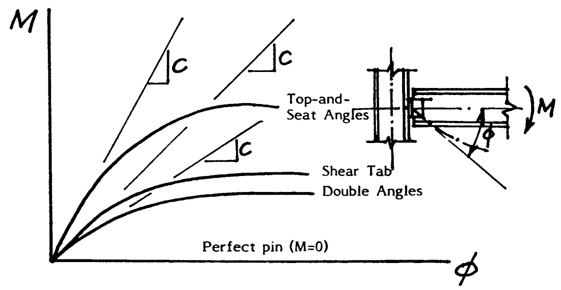

Fig. 3. Moment rotation curves for some typical simple connections (schematic)

Table 1. Sample Data for Connection Stiffnesses, C[12]

Connection Data	Value of C (K-in./rad)
Double angle, 3 bolts	3.23×10^4
Double angle, 5 bolts	28.6×10^4
Double angle, 7 bolts	90.9×10^4
6×4×¾ top and seat angles with:	
12″ beam depth	357×10^4
16″ beam depth	556×10^4
21″ beam depth	833×10^4

amount of restraint, should not the beneficial effect of this restraint be used in a further refinement of the design of the structure? Specifically, in the design of sway-prevented frames, the effects of the end restraint could be used to (a) reduce the maximum positive moment in the beams, and (b) reduce the effective length factor for the columns to something less than 1.0. The evaluation of the latter influence is the object of this paper; the effect of the relatively low restraint on the design of the beams is not perceived to be as significant. However, this topic is open for further study.

The amount of reduction of the K-factor obviously depends on the size and length of the column, as well as the type and size of the beam-to-column connection. This will be discussed in detail in the subsequent sections of this paper. However, at this stage, it is useful to demonstrate the steel weight savings that would result if $K = 0.9$ were used instead of $K = 1.0$, to choose a modest decrease in the effective length factor. Table 2 shows the column sizes (in ASTM A36 steel) that would be required for nominal column lengths of 12, 14 and 16 ft using column loads of 150 to 300 kips, and designing by the AISC *Specification.*[9] In all cases, the minimum weight shape has been chosen, and no amount of under-design ($F_a < f_a$) has been allowed, however small. It is noted the column allowable stress, F_a, in all cases has been based on weak axis buckling of the member.

The weight savings vary between zero and 11.1% for the column length and the load combinations that are given in Table 2. Admittedly a very limited set of data, the potential for structural economies is nevertheless significant.

THE STRENGTH OF COLUMNS

Before proceeding with the development of the design criteria for columns that are restrained at the ends by simple beam-to-column connections, it is necessary to detail the major column strength parameters and how they affect the behavior of the members. Extensive discussions of this subject are provided in the literature,[1,2,14] including the development of deterministic and probabilistic criteria for design curves.[1,2,14,15]

In the following evaluation only the maximum strength of columns will be considered. This is the load that corresponds to the peak of the load-deflection curve for the

Table 2. Required Column Sizes for K = 1.0 and K = 0.9

Load (kips)	$L_c = 12'$		$L_c = 14'$		$L_c = 16'$	
	K = 1.0	K = 0.9	K = 1.0	K = 0.9	K = 1.0	K = 0.9
150	W10x33	W8x31	W8x35	W8x35	W10x39	W8x35
Weight saving	—	6.1%	—	0%	—	10.3%
200	W10x45	W8x40	W8x48	W10x45	W10x49	W8x48
Weight saving	—	11.1%	—	6.3%	—	2.0%
250	W10x49	W10x49	W12x53	W10x49	W10x54	W12x53
Weight Saving	—	0%	—	7.5%	—	1.9%
300	W12x58	W12x58	W12x65	W10x60	W12x65	W12x65
Weight saving	—	0%	—	7.7%	—	0%

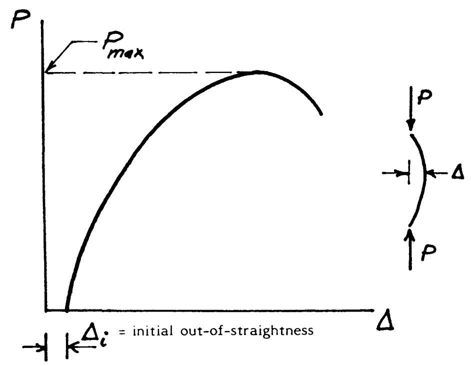

Fig. 4. Load-deflection curve for column

members, as in Fig. 4. It is emphasized that the curve in Fig. 4 is representative of columns which are initially crooked, and the maximum strength may be significantly different from that of the initially perfectly straight member, the tangent modulus load.[1] The curve is also indicative of the gradual softening of the column, as opposed to the sudden appearance of lateral deflections that pertains to the true buckling phenomenon (i.e. bifurcation of the equilibrium).

The length of a pinned-end column, or the effective length of one whose end conditions are different, has a major influence on column strength. Although increasing length does not reduce the load-carrying capacity as rapidly as predicted for elastic columns, the reduction is particularly prevalent in the intermediate range of slendernesses (L/r between 50 and 100).

The other major column strength parameters can be itemized as follows:[2]

(i) Grade of steel
(ii) Manufacturing method
(iii) Size of cross section
(iv) Shape of cross section
(v) Axis of bending
(vi) Magnitude of initial out-of-straightness
(vii) End restraint

Item vii is a relatively recent addition to the list, but purely in the sense that it is only over the past few years that detailed research has been conducted to quantify the effects. This subject will be discussed in some detail in the following section of this paper.

The importance of the grade of steel is obvious, and this is also tied to the size and shape of the cross section to the extent that the yield stress is a function of the thickness of the material. Naturally, the latter two parameters dictate the cross-sectional bending stiffness as well, and therefore indicate the relevance of the axis of bending.

The method of manufacturing is the primary influence on the magnitude and distribution of the residual stresses in the cross section. Of particular importance are the compressive residual stresses; these essentially act as a "preload" on the member, and therefore consume part of the axial load-carrying capacity of the column. Consequently it is important to know whether the shape has been produced by rolling or welding, since the level and form of the heat input that was provided control the residual stress distribution.

The magnitude and distribution of residual stresses have been studied in detail by a number of researchers,[1] as have their effects on the strength of columns.[1,2,16] In particular, the presence of compressive residual stresses lowers the column strength, especially in the intermediate slenderness ratio range. The effect is compounded by the fact that this is also the region where the initial out-of-straightness (see item vi) has the largest influence, and the two are not additive. In other words, residual stresses and initial crookedness have a synergistic influence, whereby the combined effect is more than the sum of the parts.[2] For the kind of low (10 ksi or less) residual stresses that are found in small hot-rolled shapes, the combined effect may actually be less than the sum of the parts, for certain slenderness ratios.[16] However, realistic residual stresses in actual shapes will produce a combined effect that is more unfavorable than the sum of the parts. The degree of synergism varies with the slenderness ratio.

The presence of initial out-of-straightness creates immediate bending in the column, with the result the load-deflection curve takes on the general shape indicated in Fig. 4. It has little influence on the strength of very short members, and very long columns tend to behave much like elastic, straight ones.[1,2] In the case of the former, the failure tends to be by squashing; for the long columns, the length effect is so significant that the sum of the applied and the residual stresses does not reach the yield stress before the maximum strength is achieved.

The many factors that influence the maximum strength of columns emphasize the complexity of the problem. However, computational techniques are available that allow accurate reproduction of the results of column tests. This has in large measure obviated the need for the latter, although confirmatory experiments are still conducted. Exclusive reliance on tests results is therefore fairly much a thing of the past.

INFLUENCE OF END RESTRAINT

Extensive studies of the influence of end restraint on the strength and behavior of columns have been conducted by Chen,[5,17,18] Nethercot,[4,16,19] Galambos,[3] and Razzaq,[20] among others. In addition, the analysis of frames with flexible connections has been dealt with in several studies.[12,13,21,22] The column investigations have examined different aspects of restrained member behavior; specifically, determining the influence of:

(i) Type of beam-to-column connection
(ii) Length of column
(iii) Magnitude of distribution of residual stresses
(iv) Initial out-of-straightness

The frame analysis studies have focused on evaluations of the drift characteristics of frames with less than fully rigid connections, in part prompted by a study of Disque's.[23] However, frame-related subjects of this kind are considered to be beyond the scope of this paper.

As would be expected intuitively, the stiffness of the restraining connection is a major factor. One illustration of the influence is given by the M-ϕ-curves in Fig. 3, another by the load-deflection curves for columns with different end restraint that is shown in Fig. 5.[19] A British wide-flange shape was used for the data generated for Fig. 5, incor-

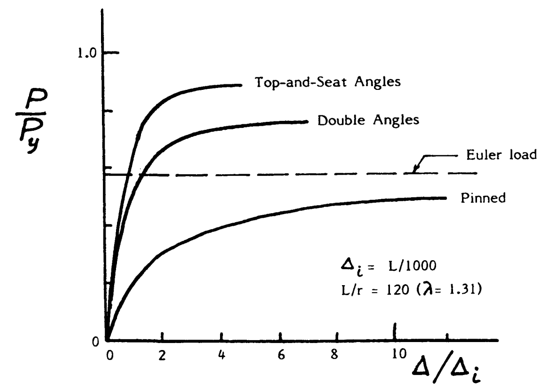

Fig. 5. Typical load-deflection curves for columns[19]

porating an initial out-of-straightness of L/1000. The curves that are shown apply for a slenderness ratio of 120 (λ = 1.31), but similar data were generated for shorter and longer columns as well.[19] Other investigators have developed additional load-deflection curves; the primary differences between the individual studies are found in the methods of column and restraint modeling, but the resulting curves are very similar.[6,18,19,20]

Figure 5 also includes the load-deflection curve for a pinned-end column. As is evident, the higher the connection restraint, the steeper will be the initial response of the column, and the higher the maximum load that can be carried. The same relative picture emerges for all slenderness ratios, although the magnitude of the increase becomes small as L/r goes towards values of 50 and less. (The increase referred to compares the strength of the restrained column to that of the pinned-end one.)

A further illustration of the influence of the end restraint is given by the data in Fig. 6, which shows column strength curves for members with a variety of end conditions.[5,19] The effect of the connection type is again evident, as is the fact that the influence diminishes for shorter columns. Also included in the figure is the Euler (=elastic buckling) curve, as well as SSRC Curve No. 2,[1,2] which represents the average maximum capacity of a large number of steel shapes.

It is emphasized that the connections used to develop the column curves in Fig. 6 are all of the simple type. The potential for the structural economies that may be gained by incorporating the end restraint into the column design procedure is evident, although the realistic ranges for the values of λ also must be borne in mind. The latter have been delineated in Fig. 6 for steels with yield stresses of 36 and 50 ksi. Consequently, the very large column strength increases that have been reported by several researchers are real enough,[4,5,18,19,20,24] but they occur for slenderness ratios that are well in excess of practical values.[25,26]

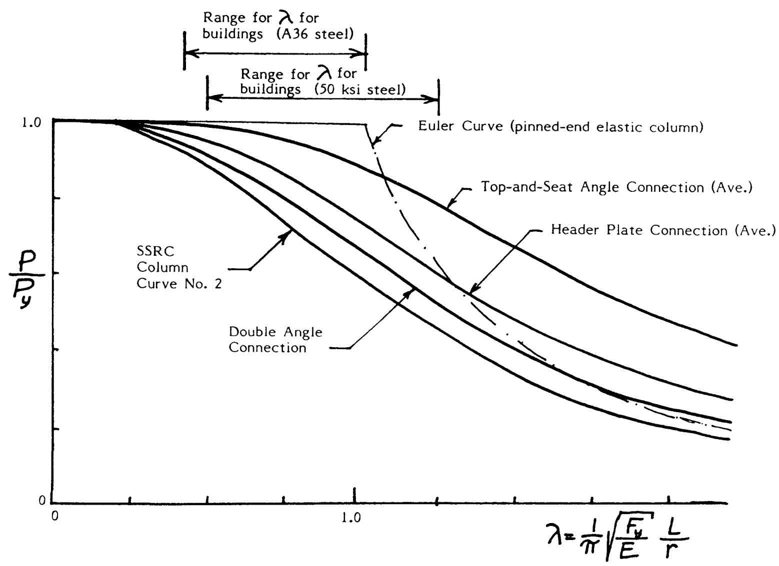

Fig. 6. Column curves for members with different types of end restraint

Overall, the end restraint has a clear strength-increasing effect. It is more pronounced for columns without residual stresses than for those where they have been included, and also more pronounced for initially crooked columns with increasingly stiff connections. Although realistic connections and ranges of slenderness ratios may not produce as significant results as some research data show, the structural advantages can be important. In other words, sufficient data are available to quantify the influence of end restraint. The subsequent sections of the paper will demonstrate how the concepts may be put to practical use.

PRACTICAL DESIGN WITH END RESTRAINT

In the design of a column as part of a frame, the basic frame geometry will have been determined at an early stage to the extent that story heights, L_c, and beam or girder spans, L_g, are known. The girder loads are computed, and these members are then sized on the basis of their moment distribution. Refinements may be introduced in the beam design at a later stage, but in the case of sway-prevented structures such as that illustrated in Fig. 1, the size of the beam is almost always governed by the gravity loads (dead and live loads). In other words, in the traditional simply supported beam design, the loads and the span of the beam dictate the size of the cross section. (Naturally, the steel grade is important, but that is assumed to be given along with the loads, etc.)

Strictly speaking, in the same fashion that end restraint can be taken into account in the design of compression members, so it also can for beams. Making use of the fact that any connection that possesses rotational restraint will reduce the positive moments in the beam, this reduction can now be applied to the governing positive bending moment. Figure 7 gives a schematic demonstration of this

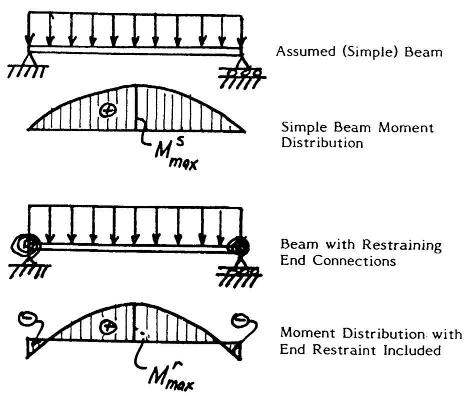

Fig. 7. Using effect of end restraint in beam or girder design for frame

principle. This topic will not be addressed in this paper, but it has some obvious practical applications, and a detailed investigation would seem to be warranted. Philosophically, it should be applicable both in allowable stress and ultimate strength design. In the case of the former, the restraining moment could be set equal to $\phi_a C$ or a fraction thereof, where ϕ_a is the connection rotation at allowable load. In the case of ultimate strength design, the restraining moment would equal the ultimate moment capacity of the connection.

The straight multiplication of ϕ_a with C, the initial response of the connection, would appear to be unconservative, as shown in Fig. 8. However, it is important the actual

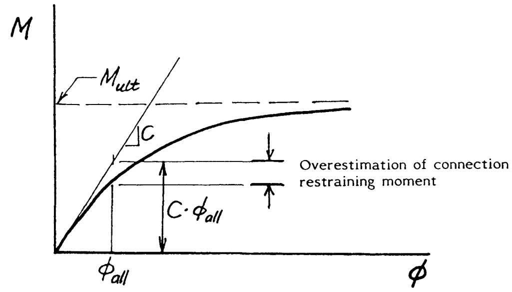

Fig. 8. Restraining moments for connection for beam or girder design in frame

magnitudes of ϕ_a be kept in mind: With a simply supported beam end rotation of $wL_g^3/24EI_g$, where w is the allowable uniformly distributed load, E = modulus of elasticity and I_g = beam moment of inertia, a typical value of ϕ_a would be 0.0092 radians (laterally supported W18x50, A36 steel, 25-ft span, fully uniformly loaded). With a common simple connection for this beam, the degree of overestimation of the restraining moment would be very small (less than 2% of the maximum positive moment in the beam).

Returning to the column design problem, with the beam size determined and the end reaction known, the type and size of the beam-to-column connection are then chosen and readily designed, respectively. For example, in the case of the W18x50 beam used above, the allowable end reaction is 28.4 kips, which requires a minimum of 2 or 3¾-in. A325 bolts in a double-angle connection, depending on the size of angles used. This information can now be used to determine the moment-rotation curve for the connection (or rather, the C-value) on the basis of published data. Some sample data are shown in Table 1; others are available in various papers and reports, and should be compiled in a single document. Studies have also been done to develop standardized M-ϕ-curves, notably the research of Frye and Morris[13] and Nethercot el al.[19]

Although connection stiffness data are not currently available for all conceivable types of simple connections, much has been generated, and missing properties can (and must) be developed. However, proceeding on the assumption that the C-value of the chosen connection is available, the design of the column can now proceed with the help of the data generated by Lui and Chen,[5,17] who, in turn, relied on much of the earlier work of the author of this paper.[2,14] Analyzing the strength of 83 end-restrained columns, Lui and Chen found that the effective length factor could be simply expressed as[17]

$$K = 1.000 - 0.017\,\alpha \quad \text{for } 0 \le \alpha \le 23 \qquad (1a)$$

and

$$K = 0.600 \text{ for } \alpha > 23 \qquad (1b)$$

where the term α is the relative stiffness factor, defined as

$$\alpha = \frac{C}{M_{pc}} \qquad (2)$$

and M_{pc} = plastic moment capacity of the column shape.

In consequence, once the connection type and size have been determined and a trial size chosen for the column, the relative stiffness factor can be computed by Eq. 2 and the K-factor by Eqs. 1a or 1b. The design of the individual member now reverts to well-established procedures, since KL can be found, and the allowable stress (or ultimate stress, for load and resistance factor design, e.g.) may be determined.

A simple design example will be used to demonstrate the design procedure.

Given:
Column length: 14′
Steel grade: A36
Connection: 2L with 3¾-in. A325 bolts,
i.e. $C = 3.23 \times 10^4$ k-in./rad
Axial load: 200 kips
Try a column W8x48 (required for $K = 1.0$)
For weak axis bending of the W8x48, the plastic moment capacity is

$$M_{pc} = Z_y F_y = 22.9 \times 36 = 824 \text{ k-in.}$$

Relative stiffness factor (Eq. 2):

$$\alpha = \frac{C}{M_{pc}} = \frac{3.23 \times 10^4}{824} = 39.2$$

Since $\alpha > 23$, Eq. 1b gives a K-value of

$$K = 0.6$$
$$\text{and } KL = 0.6 \times 14 = 8.4 \text{ ft}$$

Allowable load for the W8x48:
$P_{all} \approx 260$ kips

Redesign with a column shape W8x40:

$$M_{pc} = 18.5 \times 36 = 666 \text{ k-in.}$$
$$\alpha = \frac{3.23 \times 10^4}{666} = 48.5 > 23$$

i.e. $K = 0.6$
i.e. $P_a \approx 215$ kips
Steel weight savings: 16.7%

It is noted that Lui and Chen[17] suggest that a simpler and more conservative approach be taken, specifically, by recommending certain fixed K-values that depend on the axis of bending as well as the slenderness ratio. The recommendation is made on the premise that column design that involves the rotational stiffness of the connection will be too complicated. Although a reasonable assumption, the K-values that are suggested (varying between 0.9 and 1.0) are too conservative. Furthermore, in the design of columns as parts of frameworks, the relative stiffness factor is influenced by the stiffness of the beams that are connected to the column. As will be shown, this leads directly to a design procedure that makes use of the well-known approaches of the K-factor alignment charts, and which includes the effect of inelastic column behavior. In other words, the degree of conservatism of the Lui and Chen suggestion is not necessary, despite the fact the design approach does become somewhat more complicated. However, as will be shown by the method of column design developed for frames, the Lui and Chen method may be unconservative in some cases, especially when exterior columns are considered.

INFLUENCE OF BEAM STIFFNESS

The evaluation that has been provided so far has been based on the behavior of a single column, restrained at the ends by a connection whose initial stiffness is given by the value of C. In effect, this may be illustrated by the column shown in Fig. 9a. The column shown in Fig. 9b demonstrates the other extreme, namely, where the restraining connection is attached to a perfect pin, and no buckling restraint is given. Cases a and b correspond to having the connections attached to a perfectly rigid ($EI_g = \infty$) beam and a beam whose

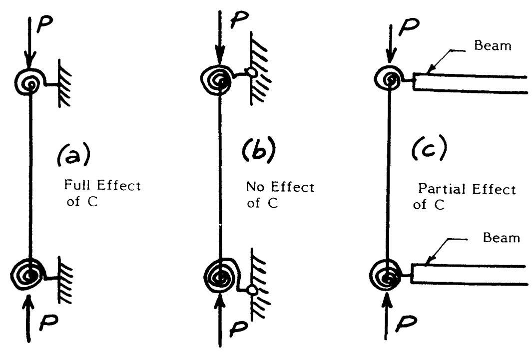

Fig. 9. Influence of support details for effective restraint

bending stiffness is equal to zero, respectively. The K-factor of case a is that which has been developed in the preceding section of the paper. The K-factor of case b is equal to 1.0, since the element supporting the connection offers no restraint to any rotation.

The real support conditions are schematically illustrated in Fig. 9c, where it is shown that the beam-to-column connection is attached to a beam of a certain bending stiffness. Using a bending rigidity of EI_g and a beam span of L_g, the bending stiffness of the elementary, simply supported beam that is shown in Fig. 10a is given by

$$\text{stiffness} = M_{\phi=1} = \frac{2EI_g}{L_g} \tag{3}$$

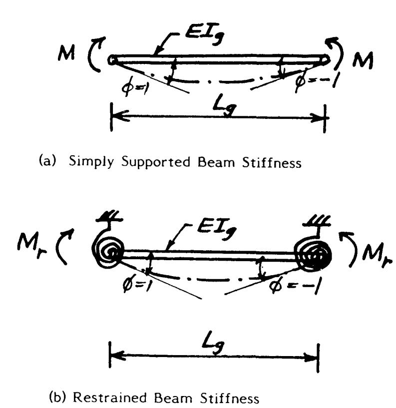

Fig. 10. Stiffness of simple and restrained beams

and the assumptions regarding deflection shape and end rotations are indicated in the figure.

For a beam that is restrained at both ends by a connection with a certain C-value, and otherwise using the same principles and rotation senses and magnitudes, the bending stiffness is given by[17]

$$\text{stiffness} = M_r = C^* = \frac{2EI_g}{L_g} \frac{1}{1 + (2EI_g)/(L_g C)} \quad (4)$$

where C* is the effective end restraint that is provided for the column. In other words, the bending stiffness of the overall supporting structure, which in turn is what is reflected upon the buckling behavior of the column, is reduced from that of case a in Fig. 9 to that of case c in the same figure.

Defining the term β as the connection-to-beam factor, and the term β* as its effective value, Eq. 4 can be rewritten as

$$\beta^* = \frac{C^*}{EI_g/L_g} = \frac{2\beta}{\beta+2} \quad (5)$$

where β is given by Eq. 6 as

$$\beta = \frac{C}{(EI_g/L_g)} \quad (6)$$

As will be demonstrated subsequently, the connection-to-beam factor can be conveniently utilized in the analysis of the restrained column as part of a frame.

Figure 11 shows the relationship between β and β*. For the case when $\beta = 0$, this means that the column is connected to a pin, as shown in Fig. 9b, and the K-factor is equal to 1.0. For $\beta > 0$, the effective value of β* can be determined from Fig. 11, or simply computed from Eq. 5, and the effective connection restraint is given as

$$C^* = \beta^* \times \left(\frac{EI_g}{L_g}\right) \quad (7)$$

This value of the restraint should now be utilized in the design of the end-restrained column, calculating the effective relative stiffness by a modified form of Eq. 2, thus

$$\alpha^* = \frac{C^*}{M_{pc}} \quad (8)$$

α* is then used with Eqs. 1a or 1b to determine the effective length factor for the column, thus taking into account the more realistic support conditions that have been outlined in the preceding.

For the case when β goes toward infinity, the asymptotic value of β* is 2.0, as indicated in Fig. 11. This corresponds to the behavior of a column that is rigidly attached to the beam, and the beam is bending in the symmetric mode illustrated by Fig. 10a. In other words, $C^* = 2EI_g/L_g$, which is the correct solution for a column that is rigidly attached to a beam.

The importance and ease of application of the above principles will be further emphasized by their incorporation into frame analysis methods. This will be discussed in the next section of the paper. At this stage the use of the effective rotational restraint and relative stiffness factors will be demonstrated by expanding the example given in the section "Practical Design with End Restraint."

Given:

Column length: 14′
Steel grade: A36
Axial load: 200 kips
Connection: 2L with 3¾-in. A325 bolts,
i.e. $C = 3.23 \times 10^4$ k-in./rad
Beam (same at top and bottom): W18x50; 25′ span

Try a column W8x40 (same as final choice in example in the section "Practical Design with End Restraint"). The weak axis plastic moment capacity is

$$M_{pc} = 666 \text{ k-in.}$$

Value of (EI_g/L_g) for the beam:

$$EI_g/L_g = [(29{,}000) \times (800)]/(25 \times 12) = 7.73 \times 10^4$$

Connection-to-beam factor:

$$\beta = \frac{C}{(EI_g/L_g)} = \frac{3.23 \times 10^4}{7.73 \times 10^4} = 0.42$$

Effective β-value (by Eq. 5):

$$\beta^* = \frac{2\beta}{\beta+2} = \frac{2 \times 0.42}{0.42+2} = 0.35$$

Effective connection rotational stiffness (Eq. 7):

$$C^* = \beta^* \times \left(\frac{EI_g}{L_g}\right) = 0.35 \times 7.73 \times 10^4 = 2.71 \times 10^4$$

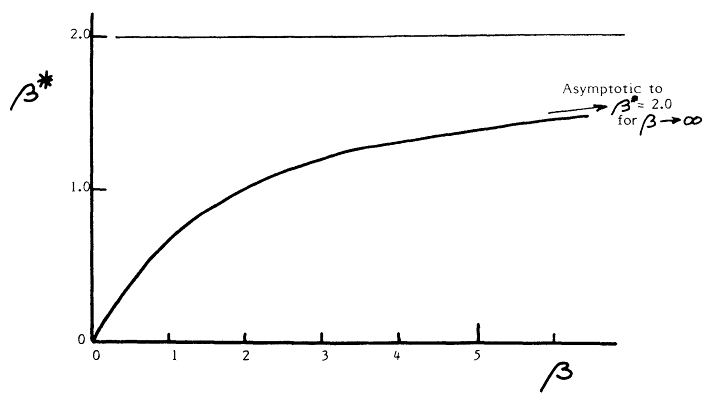

Fig. 11. Effective vs. initial connection-to-beam factor

Effective relative stiffness (Eq. 8):

$$\alpha^* = \frac{C^*}{M_{pc}} = \frac{2.71 \times 10^4}{666} = 40.6$$

Since $\alpha^* > 23$, Eq. 1b gives a K-factor of
$K = 0.600$

and the W8x40 therefore continues to be satisfactory for the intended usage.

It is observed the procedure outlined assumes the connections and the beams are identical at the top and the bottom ends of the member. The method of allowing for differences will be presented in the section "The Column as Part of the Frame."

It is also noted the beam deflection shape used to determine the influence of the beam stiffness, as shown in Figs. 10a and b, produces the least amount of restraint for the column. For far end support conditions that are fixed ($\phi = 0$), or with a moment that produces a complete anti-symmetric deflection shape, the beam stiffness will be larger than that expressed by β^*. The approach used is therefore conservative for the individual member. The following section will address the application of column end restraint in the overall frame design.

THE COLUMN AS PART OF A FRAME

The real support conditions for the column are naturally those of the structure, where it is connected to beams or girders, other columns (as in the case of continuous members), or to column bases. The original stability solution for this problem was provided by the Julian and Lawrence nomographs,[7] which incorporated both the sway-prevented and the sway-permitted frames. A frame subassemblage was used to formulate the solution; the sway-prevented case and its assumed buckling shape are shown in Fig. 12.

A number of assumptions had to be made by the originators of the alignment charts, and studies have since been conducted to evaluate the influence of some of these criteria.[8,27] Of particular interest to the current evaluation is the fact that the columns in a given story had to fail simultaneously. Significant modifications have since been introduced, notably the concept of the inelastic K-factor, developed by Yura[27] and further refined by Disque.[28] This technique has since been incorporated into the AISC Manual,[11] and is particularly useful for the design of columns in sway-permitted frames. However, it can also be applied to sway-prevented structures, as will be demonstrated in the following.

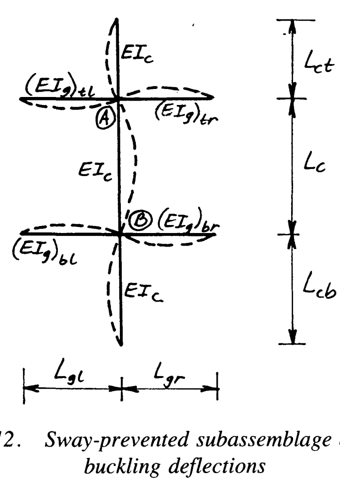

Fig. 12. Sway-prevented subassemblage and its buckling deflections

The characteristic equation (buckling condition) for the subassemblage shown in Fig. 12[1] is:

$$\left(\frac{G_A G_B}{4}\right)\left(\frac{\pi}{K}\right)^2 + \left(\frac{G_A + G_B}{2}\right)\left(1 - \frac{\pi/K}{\tan(\pi/K)}\right) + 2\,\frac{\tan(\pi/2K)}{(\pi/K)} = 1 \quad (9)$$

where G^A and G^B are the stiffness distribution factors for column ends A and B, respectively. They are defined mathematically as

$$G = \frac{\Sigma(EI_c/L_c)}{\Sigma(EI_g/L_g)} \quad (10)$$

where the summation in the numerator covers the stiffnesses of the columns that come together at a joint, and the summation in the denominator covers the stiffnesses of the beams at the same joint. G expresses the relative stiffness of the columns vs. the beams at a joint, and, in general, the higher the value of G, the more moment the column will be asked to carry. In the extreme, the columns at a joint are infinitely stiff as compared to the beams, which simply means the beams transfer no moment to the column; or put differently, cannot stabilize the column against buckling. In other words, $G = \infty$ indicates a pinned end for the column.

In the common design approach for a sway-prevented frame, the columns are conservatively designed with a K-value of 1.0. They are consequently treated as if the beams are attached through perfect pins. In reality, the beams (with a certain span and bending rigidity) are attached to the column with connections that have a certain rotational restraint. The total restraint that is afforded by this support condition is none other than the effective rotational restraint, C^*, defined by Eq. 7 and developed through Eqs. 5 and 6.

On the basis of the philosophical concept of the influence of the effective end restraint for a column, the stiffness distribution factor of Eq. 10 can now be rephrased as it applies to a column of a sway-prevented structure, thus:†

$$G_r = \frac{\Sigma(EI_c/L_c)}{C^*} \quad (11)$$

†The principles utilized here also make it possible to extend the G_r concept to a sway-permitted structure, as long as the appropriate C-values are used. However, this will not be discussed in this paper.

where G_r is the stiffness distribution factor for an end-restrained system. The use of a single C* as opposed to the summation of end restraints at a joint will be explained in the following.

In the design of end-restrained columns in frames, the inelastic K-factor procedure[27,28] should be utilized. As detailed in the AISC Manual,[11] this involves computing the inelastic value of G, which is a function of the term f_a/F'_e. The inelastic G-value is then used with the K-factor alignment chart to find the inelastic K. The approach will be demonstrated in a design example later in this section.

The behavior of columns, beams and connections before and during column buckling is important to the effective restraint that is afforded the column. It is also necessary to distinguish between exterior and interior columns. Figure 13 illustrates these aspects as they apply to interior columns (Fig. 13a) and exterior columns (Fig. 13b).

(a) Behavior of Interior Columns

Deflection curves of the beams before column buckling are indicated by dash-dot lines. All four beam-to-column connections are loaded and rotated a certain amount. As the column buckles, and therefore assumes a deflected shape, certain connections will continue to rotate, and others will have their prebuckling rotation reduced. Specifically, at the top of the column in Fig. 13a, connection (L) will load, and connection (R) will unload. At the bottom of the column, connection (L) will unload, and connection (R) will load.

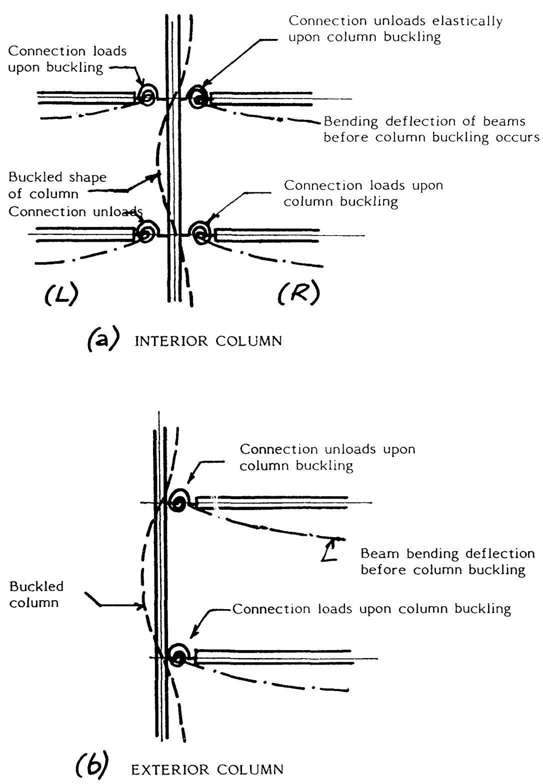

Fig. 13. Buckling behavior of end-restrained columns in sway-prevented frame

The loading and unloading responses of the connection are demonstrated by the M-ϕ-curve in Fig. 14. The prebuckling rotation of the connection equals ϕ_b, corresponding to the end rotation of the beam under gravity load. If additional rotation is imposed on the connection, its rotational stiffness at this stage equals C_t, the tangent stiffness. Depending on the shape of the curve and the magnitude of ϕ_b, C_t may be small. It is therefore conservative to say that its rotational stiffness may be neglected. By the same token, a connection that is being unloaded from the point (M_b,ϕ_b) on the M-ϕ-curve will unload elastically. Its rotational restraint is therefore equal to C, the initial slope of the curve.

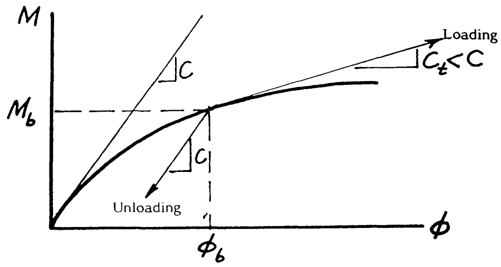

Fig. 14. Loading and unloading characteristics of connections

In consequence, it is seen that one connection (and beam) at each end of the column will afford rotational restraint corresponding to the initial value. In other words, the overall restraint effect of the beam and the beam-to-column connection equals one C* at each end, and thus develops the expression for G_r in Eq. 11.

(b) Behavior of Exterior Columns

The prebuckling beam deflection curves and the buckled column shape are shown in Fig. 13b. As buckling occurs, the connection at the top will unload, and the connection at the bottom will load. The top restraint is therefore given by C*; the bottom restraint can be conservatively set equal to zero.

The exterior column therefore should be designed on the basis of using G_r for one end and a pin at the other ($G = \infty$). The sway-prevented nomograph may then be used to find the K-factor. Naturally, the inelastic K-value procedures should be followed.

Design Example:
Interior column axial load: 200 kips
Column length: 14 ft
Steel Grade: A36
Connections: 2L with 3¾-in. A325 bolts,
i.e. $C = 3.23 \times 10^4$ k-in./rad

Beam sizes: W18x50 with 30 ft span on right side; W16x45 with 25 ft span on left side

Try W8x40 for column (bent about weak axis):

$M_{pc} = 666$ k-in.

(EI_g/L_g) for W18x50 $= [29{,}000 \times 800/30 \times 12] = 6.44 \times 10^4$

(EI_g/L_g) for W16x45 $= [29{,}000 \times 586/25 \times 12] = 5.66 \times 10^4$

Connection-to-beam factors:

$$\text{W18x50: } \beta = \frac{3.23 \times 10^4}{6.44 \times 10^4} = 0.50$$

$$\text{W16x45: } \beta = \frac{3.23 \times 10^4}{5.66 \times 10^4} = 0.57$$

Effective β-values:

$$\text{W18x50: } \beta^* = \frac{2 \times 0.50}{0.50 + 2} = 0.40$$

$$\text{W16x45: } \beta^* = \frac{2 \times 0.57}{0.57 + 2} = 0.44$$

Effective connection restraint values:

W18x50: $C^* = 0.40 \times 6.44 \times 10^4 = 2.58 \times 10^4$

W16x45: $C^* = 0.44 \times 5.66 \times 10^4 = 2.49 \times 10^4$

Column stiffness values:

$(EI_c/L_c) = [29{,}000 \times 146/14.12] = 2.52 \times 10^4$

The same column size is assumed to continue above and below the column that is being designed (column AB in Fig. 12).

Elastic stiffness distribution factors:

$$\text{Top: } G_r = \frac{\Sigma(EI_c/L_c)}{C^*} = \frac{2 \times 2.52 \times 10^4}{2.58 \times 10^4} = 1.95$$

$$\text{Bottom: } G_r = \frac{2 \times 2.52 \times 10^4}{2.49 \times 10^4} = 2.02$$

Determine inelastic characteristics:

$$f_a = \frac{P}{A} = \frac{200}{11.7} = 17.09 \text{ ksi}$$

From Table A, p. 3-7 of the AISC Manual,[11] this gives a value of f_a/F'_e of

$$f_a/F'_e = 0.461$$

$$(G_r)_{\text{inelastic}} = G_r \times \frac{f_a}{F'_e} = 1.95 \times 0.461 = 0.90 \text{ (top)}$$

and

$$(G_r)_{\text{inelastic}} = 2.02 \times 0.461 = 0.93 \text{ (bottom)}$$

Determine the K-factor:

Using the sway-prevented alignment chart with the inelastic G_r-data as found above,

$K = 0.76$

This gives an effective length of

$KL = 0.76 \times 14 \times 10.6'$

Determine the allowable load:

The allowable load for this length of a W8x40 in A36 steel is:

$P_a \approx 202$ kips

The member is therefore satisfactory.

The above example and that given in the section "Influence of Beam Stiffness" are not fully comparable, because of the additions of more beams (W16x45) at both column ends of the former. On the basis of the individual column design procedure of that section, this would produce larger C*-values for the member, resulting in a reduced K-factor (unless it is already at the minimum K of 0.6, as is the case for that section's example). However, it is very important to observe that even with the increased end restraint, the K-factor for the column as determined from the stability solution for the frame subassemblage, is actually larger than that obtained on the basis of the individual member treatment. This is a result of the fact that only a single C*-value restrains the column at the ends, as well as the fact that it is governed by the overall stability solution for the frame subassemblage.

In consequence of the above observations, the individual column design procedure, as originally developed by Lui and Chen,[17] may produce somewhat unconservative (i.e., too small) K-factors, especially for exterior columns. Although the actual effect of this difference may be relatively small, depending on the slenderness ratio of the member and the type of column curve that is used, in a practical frame design the procedure that is outlined under B., Design of Columns in Frames, should be utilized.

RECOMMENDED DESIGN PROCEDURE

The theoretical developments that have been presented along with the detailed design examples illustrate the practical application of design of columns with end restraint. The sequence of the design steps can now be summarized as follows:

A. Design of Individual Columns

(a) With frame geometry and loads defined, size beams on the basis of simple span behavior

(b) End reaction from (a) determines a certain connection shear capacity. Choose connection type; design connection for end reaction.

(c) Connection type and size give the value of C, the rotational end restraint.

(d) Compute $\alpha = C/M_{pc}$, using M_{pc} for the trial column size and framing direction.

(e) With beams framing into column, compute (EI_g/L_g), β and β^* (Eq. 5).

(f) Compute effective connection rotational restraint, C^* (Eq. 7).
(g) Compute effective relative stiffness (Eq. 8), α^*.
(h) Use α^* in Eqs. 1a or 1b to find the effective length factor, $K \leq 1.0$ for sway-prevented frames.
(i) Determine allowable load, P_a.
(j) Try new column shape if the allowable load is less than the required load, or if the allowable load leaves too much unused capacity.

B. Design of Columns in Frames

(a) Repeat steps a through c, above.
(b) Compute (EI_g/L_g) for beams at top (right and left) and at bottom (right and left) of the column.
(c) Compute β-values for the top (R and L) and the bottom (R and L) of the column.
(d) Compute effective β-values, β^*.
(e) Compute effective connection restraint values, C^*.
(f) Select a trial column size.
(g) Compute column stiffness, (EI_c/L_c).
(h) Compute elastic stiffness distribution factors, G_r. *For interior columns,* use the smallest C^* at the top and at the bottom of the column (conservative approach) in the denominators for the respective G_r-values.
For exterior columns, use $G_r = \infty$ at top (or at bottom), and use the smallest C^* in the computation of G_r for the other end of the column.
(i) Compute the applied axial stress, f_a, and use this to find f_a/F'_e (AISC Manual[11]).
(j) Compute inelastic G_r - values.
(k) Use the alignment chart for sway-prevented frames with the inelastic G_r - values to find the effective length factor, $K \leq 1.0$.
(l) Determine the allowable load, P_a.
(m) Redesign, if necessary.
Note: For end supports, use $G = 10$ or 1, depending on whether base is pinned or fixed.

SUMMARY AND CONCLUSIONS

A design procedure has been developed for columns in sway-prevented frames, whereby the influence of connection end restraint can be taken into account explicitly. It is shown that the current, traditional approach of using an effective length factor of $K = 1.0$ for the columns in such frames tends to be overly conservative. The method that is developed takes into account the combined restraining effect of the connection and the beam to which it is attached, recognizing that the beam flexibility tends to reduce the full restraint effect of the connection.

The design procedure is extended to a frame stability analysis, using the familiar nomographs for K-factors, but redefining the stiffness distribution factors in terms of column stiffness and connection-and-beam restraint. It is demonstrated how the behavior of interior and exterior columns dictate different treatment. It is also shown that the individual member design approach tends to produce somewhat unconservative results, especially for exterior columns. This is a result of the overall stability solution for the end-restrained column as part of a subassemblage. The method outlined in the section, "The Column as Part of a Frame" and detailed in the procedure in the section, "Recommended Design Procedure," Part B should therefore be utilized in practical frame design.

Detailed design examples are given throughout the paper, and the economies of the proposed design method are unmistakable. Preliminary results indicate potential material savings of 15 to 20%, but these figures will be analyzed in continuing studies.

ACKNOWLEDGEMENTS

Special thanks are due Robert O. Disque of AISC for the many fruitful discussions he and the author have had on the subjects of column stability and design. His sense of the practical, while maintaining a thorough appreciation of concepts and theory, has been invaluable to the work of the author.

Thanks are also due Margaret Stalker and Ann Edwards for their patience and efforts in typing the manuscript, as well as Mun-Foo Leong for performing some of the computation work.

REFERENCES

1. *Johnston, B. G.,.Ed.* Guide to Stability Design Criteria for Metal Structures *3rd Ed., Wiley-Interscience, New York, 1976.*
2. *Bjorhovde, R.* Deterministic and Probabilistic Approaches to the Strength of Steel Columns *Ph.D. dissertation, Lehigh University, Bethlehem, Pa. 1972.*
3. *Chapuis, J. and T. V. Galambos* Restrained Crooked Aluminum Columns *Journal of the Structural Division, ASCE, Vol. 108, No. ST3, March 1982 (pp. 511–524).*
4. *Jones, S. W., P. A. Kirby, and D. A. Nethercot* Columns with Semi-Rigid Joints *Journal of the Structural Division, ASCE, Vol. 108, No. ST2, February 1982 (pp. 361–372).*
5. *Lui, E. M. and W. F. Chen* Strength of H-Columns with Small End-Restraints *Structural Engineering Report No. CE-STR-81-38 (1981), Purdue University, West Lafayette, Ind., November 1981.*
6. *Jones, S. W., P. A. Kirby, and D. A. Nethercot* Effect of Semi-Rigid Connections on Steel Column Strength *Journal of Constructional Steel Research, London, England, Vol. 1, No. 1, September 1980 (pp. 38–46).*
7. *Julian, O. G., and L. S. Lawrence* Notes on J and L Nomograms for Determination of Effective Lengths *Unpublished report, Jackson and Moreland Engineers, Boston, Mass., 1959.*
8. *Kuhn, G. R.* An Appraisal of the Effective Length Alignment Charts *M.S. thesis, Arizona State University, Tempe, Ariz., May 1976.*
9. *American Institute of Steel Construction* Specification for the Design, Fabrication and Erection of Steel Structures for Buildings *Chicago, Ill., November 1978.*

10. *Lewitt, C. W., E. Chesson and W. H. Munse* Restraint Characteristics of Flexible Riveted and Bolted Beam-to-Column Connections *University of Illinois Engineering Experiment Station Bulletin No. 500, Urbana, Ill., January 1969.*
11. *American Institute of Steel Construction* Manual of Steel Construction *8th Ed., Chicago, Ill., 1980.*
12. *DeFalco, F. and F. J. Marino* Column Stability in Type 2 Construction *AISC Engineering Journal, Vol. 3, No. 2, April 1966, Chicago, Ill. (pp. 67–71).*
13. *Frye, M. J. and G. A. Morris* Analysis of Flexibly Connected Steel Frames *Canadian Journal of Civil Engineering, Vol. 2, No. 3, September 1975 (pp. 280–291).*
14. *Bjorhovde, R.* The Safety of Steel Columns *Journal of the Structural Division, ASCE, Vol. 104, No. ST3, March 1978 (pp. 463–477).*
15. *Bjorhovde, R. and P. C. Birkemoe* Limit States Design of HSS Columns *Canadian Journal of Civil Engineering, Vol. 6, No. 2, June 1979 (pp. 276–291).*
16. *Batterman, R. H. and B. G. Johnston* Behavior and Maximum Strength of Metal Columns *Journal of the Structural Division, ASCE, Vol. 93, No. ST2, April 1967 (pp. 205–230).*
17. *Lui, E. M. and W. F. Chen* End Restraint and Column Design Using LRFD *Structural Engineering Report No. CE-STR-82-24 (1982), Purdue University, West Lafayette, Ind., 1982.*
18. *Sugimoto, H. and W. F. Chen* Small End Restraint Effects on Strength of H-Columns *Journal of the Structural Division, ASCE, Vol. 108, No. ST3, March 1982 (pp. 661–681).*
19. *Jones, S. W., P. A. Kirby and D. A. Nethercot* Modelling of Semi-Rigid Connection Behavior and Its Influence on Steel Column Behavior *Proceedings, Conference on Joints in Structural Steelwork, Teeside, England, April 1981, Pentech Press, Ltd.*
20. *Razzaq, Z. and J. G. Chang* Partially Restrained Imperfect Columns *Proceedings, Conference on Joints in Structural Steelwork, Teeside, England, April 1981, Pentech Press, Ltd.*
21. *Ackroyd, M. H.* Non-Linear Inelastic Stability of Flexibly-Connected Plane Steel Frames *Ph.D. dissertation, University of Colorado, Boulder, Colo., 1979.*
22. *Romstad, K. M. and C. V. Subramanian* Analysis of Frames with Partial Connection Rigidity *Journal of the Structural Division, ASCE, Vol. 96, No. ST11, November 1970 (pp. 2,283–2,300).*
23. *Disque, R. O.* Directional Moment Connections—A Proposed Design Method for Unbraced Steel Frames *AISC Engineering Journal, Vol. 12, No. 1, 1975, Chicago, Ill. (pp. 14–18).*
24. *Chen, W. F.* End Restraint and Column Stability *Journal of the Structural Division, ASCE, Vol. 106, No. ST11, November 1980 (pp. 2,279–2,295).*
25. *Bjorhovde, R.* Discussion of "End Restraint and Column Stability" by W. F. Chen *Journal of the Structural Division, ASCE, Vol. 107, No. ST8, August 1981 (pp. 1,696–1,700).*
26. *Ackroyd, M. H. and R. Bjorhovde* Effect of Semi-Rigid Connections on Steel Column Strength: Discussion *Journal of Constructional Steel Research, London, England, Vol. 1, No. 3, May 1981 (pp. 48–51).*
27. *Yura, J. A.* The Effective Length of Columns in Unbraced Frames *AISC Engineering Journal, Vol. 8, No. 2, April 1971, Chicago, Ill. (pp. 37–42).*
28. *Disque, R. O.* Inelastic K-Factor in Column Design *AISC Engineering Journal, Vol. 10, No. 2, April 1973, Chicago, Ill. (pp. 33–35).*

NOMENCLATURE

C	= initial stiffness of a connection
C^*	= effective stiffness of a connection
C_t	= tangent stiffness of a connection (= f(ø))
E	= modulus of elasticity (= 29,000 ksi)
F_a	= allowable column axial stress
F'_e	= Euler stress divided by the factor of safety
f_a	= applied axial stress
G, G_A, G_B	= stiffness distribution factors
G_r	= stiffness distribution factor for a joint restraint
I_c	= column moment of inertia
I_g	= beam or girder moment of inertia
K	= effective length factor
L, L_c	= column length; story height
M	= moment
$M^r{}_{max}$	= maximum positive girder moment; end restraint included
$M^s{}_{max}$	= maximum positive girder moment; simple supports
M_{ult}	= ultimate moment capacity of a connection
M_{pc}	= plastic moment capacity of the column
P	= column axial load
P_a	= allowable axial load for a column
P_{max}	= maximum strength of a column
r	= radius of gyration
w	= allowable uniformly distributed load
Z_y	= weak axis plastic section modulus
α	= ratio of connection stiffness and column plastic moment capacity (= C/M_{pc})
α^*	= effective α-value
β	= connection-to-beam factor
β^*	= effective β-value
	= deflection
$_i$	= initial out-of-straightness of a column
$ø$	= connection rotation angle (radians)
$ø_a$	= connection rotation angle at allowable load

DISCUSSION

Effect of End Restraint on Column Strength — Practical Applications

Paper by REIDAR BJORHOVDE
(1st quarter, 1984)

Discussion by **Kurt H. Gerstle**

Bjorhovde's article in the AISC *Engineering Journal,* 1st Qtr. 1984 — "Effect of End Restraint. . ." — is welcome because it sheds light on the importance of choosing a correct effective column length for proper member sizing; and because it emphasizes the effect of connection stiffness on the end restraint of columns. Obviously, stiffer girder-column connections provide greater resistance to column end rotation and therefore lead to shorter effective column length.

However, you cannot have your cake and eat it too. If connection stiffness serves to restrain rotation of the column ends, then by the same token it will also keep the girder end from freely rotating, and the resulting girder end moment will have to be resisted, at least in part, by the columns. This means the column will have to be designed not as an axially loaded column, but as a beam column according to Formula 1.6-1 of the AISC Specification.

This column moment will be more prominent in the exterior than in the interior columns of a building frame, because the girders frame into only one side of the column. To explore the importance of this effect, we consider here a situation as shown in Fig. 13b of the author's paper, using the following specifications from the author's design example on p. 10, and the recommended procedure for "Design of Columns in Frame" (p. 12): Axial column load: 200 kips; column length: 14 ft. Column in weak-axis bending. Steel grade: A36. Connection stiffness: $C = 3.23 \times 10^4$ kip-in./radian. Uniform girder load: 1.5 kip/ft. Girder: W18×50, girder length: 30 ft.

For comparison, we will size the column both excluding and including the moments:

Kurt H. Gerstle is Professor of Civil Engineering, University of Colorado, Boulder, Colorado.

A. Neglecting Moments

Assume W10×45 column in weak-axis bending:

$$f_a = P/A = 15.0 \text{ ksi}$$

Girder stiffness (Bjorhovde, Eq. 4):

$$C^* = \frac{2EI}{L} \times \frac{1}{1 + 2\left(\dfrac{EI}{CL}\right)}$$

$$= .4\ EI/L = .889E$$

Column stiffness: $4EI/L = 1.27E$

$$G_{Top}^{Elas.} = 2(4EI/L)/C^* = 2.8$$

$$G_{Top}^{Inel.} = f_a/F'_e \times G_{Top}^{Elas.} = .56 \times 2.8 = 1.6$$

$$G_{Bot} = \infty$$

From Nomograph, braced frames

$K = .90$

$KL = 12.6$ ft

From AISC Table 3-27

For W10×45: $P_{all} = 208$ kips

Use W10×45

B. Considering Moments

Assume W10×49 column in weak-axis bending:

$f_a = 13.9$ ksi

$$G_{Top}^{Elas.} = 2(4EI/L)/C^* = 5.0$$

$$G_{Top}^{Inel.} = (13.9/41.5) \times 5.0 = 1.7$$

$$G_{Bot} = \infty$$

From Nomograph, braced frames

$K = .90$

$KL/r = 60$

$F_a = 17.43$ ksi

Fixed-end Girder Moments:

A design aid for the calculation of fixed-end moments of flexibly-connected girders is contained in Ref. 1.

From Fig. 5 of this reference,

$$\text{For } \frac{EI}{CL} = \frac{1}{\beta^*} = 2.0$$

$$M^F = .20\ WL^2/12 = 22.5 \text{ kip-ft}$$

Distribution of fixed-end moment to adjacent members:

Stiffness of flexibly connected girder,

$C^* = .889\ E$

Stiffness of each column $= 4EI/L = 2.22\ E$

Distribution of fixed-end moment to one column

$$= \frac{2.22}{.889 + 2 \times 2.22} \times 22.5 = 9.4 \text{ kip-ft}$$

Carry-over moment due to girder below

$$= \frac{1}{2} \times 9.4 = 4.7 \text{ kip-ft}$$

Design column moment $= 9.4 + 4.7 = 14.1$ kip-ft

AISC Formula 1.6-1a:

$f_b = 9.1$ ksi

$F_b = .75\ F_Y = 27.0$ ksi

$C_m = .4 f_a / F'_e = .33$

$$\frac{f_a}{F_a} + \frac{f_b\, C_m}{F_b\left(1 - \dfrac{f_a}{F'_e}\right)} = .80 + .20 = 1.00$$

Use W10×49

This column size is just about the same as for pure axial loading with $K = 1.0$. For this case, the beneficial effects of column end restraint and the detrimental effects of cranked-in moments seem to cancel each other out. For interior columns, for which the girder moments balance, the beneficial effects will surely predominate.

REFERENCES

1. *Ackroyd, M.H. and K.H. Gerstle* Behavior of Type 2 Steel Frames *ASCE Journal of the Structural Division, Vol. 108, No. ST7, July 1982, p. 1,541.*

Discussion by **E.M. Lui and W.F. Chen**

The author has to be commended for presenting a simple and practical approach to the end-restrained column problem. Of particular interest is the proposed use of the non-sway alignment chart in assessing the effective length factor for columns in frames which are connected to girders by simple connections. Similar attempts to approach such problems have been reported by Defalco and Marino[12] and Driscoll.[29]

In the author's paper, the original stiffness distribution factor G, based on girders rigidly connected to columns, defined as

$$G = \frac{\Sigma(EI_c / L_c)}{\Sigma(EI_g / L_g)} \tag{1}$$

is replaced by a new stiffness distribution factor G_r, based on girders not rigidly connected to columns, defined as

$$G_r = \frac{\Sigma(EI_c / L_c)}{C^*} \tag{2}$$

where

$$C^* = \frac{2EI_g}{L_g} \frac{1}{1 + (2EI_g)/(L_g C)} \tag{3}$$

Equations 1, 2 and 3 correspond to Eqs. 10, 11 and 4 in the author's paper. C is the rotational stiffness of the connection and is depicted schematically in Fig. 3 of the paper. As C approaches infinity, the connection approaches a rigid connection. In other words, the girders are rigidly connected to the columns and so Eq. 2 should reduce to Eq. 1 as C approaches infinity. However, it can be seen that this is not the case here. As C approaches infinity, C^* beomes $2EL_g / L_g$ which, in general, is not equal to $\Sigma(EI_g/L_g)$. This discrepancy can be remedied as follows.

Refer to Fig. 15a, if a girder is rigidly connected at both ends to other structural members, the basic force-displacement (moment-rotation) relationship is

$$\begin{pmatrix} M_i \\ M_j \end{pmatrix} = \frac{2EI_g}{L_g} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} \theta_i \\ \theta_j \end{pmatrix} \tag{4}$$

However, if the same girder is connected to other structural members at both ends by semi-rigid connections having rotational stiffness C (Fig. 15b), the basic force-displacement relationship will be

$$\begin{pmatrix} M_i \\ M_j \end{pmatrix} = \frac{2EI_g}{L_g} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} \theta'_i \\ \theta'_j \end{pmatrix} \tag{5}$$

E.M. Lui is a graduate student in the School of Civil Engineering, Purdue University, West Lafayette, Indiana.

W.F. Chen is Head of the Structural Engineering Department, School of Civil Engineering, Purdue University, West Lafayette, Indiana.

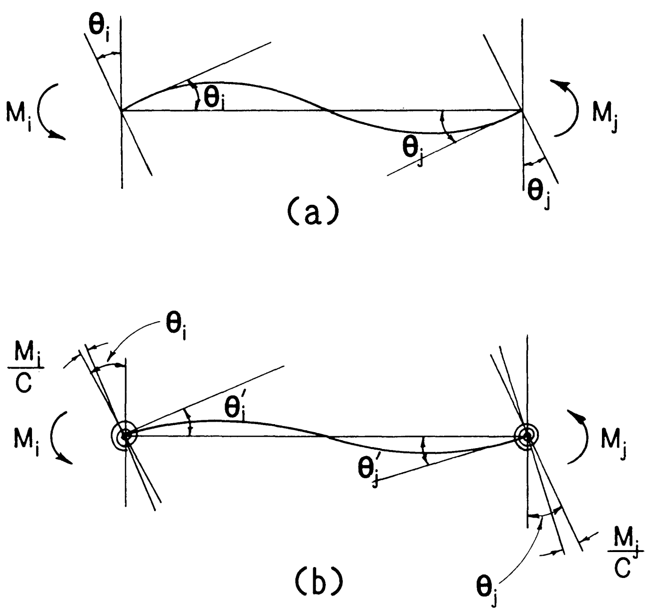

Fig. 15. Effect of semi-rigid connections on the basic force-displacement relationship of a girder

where

$$\theta_i' = \theta_i - \frac{M_i}{C} \quad \text{(6a)}$$

$$\theta_j' = \theta_j - \frac{M_j}{C} \quad \text{(6b)}$$

In writing Eqs. 4 and 5, it is tacitly assumed that there is no axial or transverse loads on the girder. This assumption is consistent with that used in the development of the alignment charts.

For non-sway frames, the alignment chart was developed under the postulation that as the column buckles, the girder will bend in single curvature with the rotation at the i-th end equal and opposite to the rotation at the j-th end. Thus, by setting $\theta_j = -\theta_i$ we can write Eq. 4 as

$$\begin{pmatrix} M_i \\ M_j \end{pmatrix} = \frac{2EI_g}{L_g}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\theta_i \quad \text{(7)}$$

and by setting the $\theta_j' = -\theta_i'$, Eq. 5 can be written as

$$\begin{pmatrix} M_i \\ M_j \end{pmatrix} = \frac{2EI_g}{L_g}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\theta_i' \quad \text{(8)}$$

Upon substitution of Eqs. 6a and 6b into Eq. 8 and solving for M_i and M_j, we can write

$$\begin{pmatrix} M_i \\ M_j \end{pmatrix} = \frac{2EI_g'}{L_g}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\theta_i \quad \text{(9)}$$

where

$$I_g' = I_g\left[\frac{1}{1 + 2EI_g/L_gC}\right] \quad \text{(10)}$$

Comparison of Eqs. 7 and 9 shows that the effect of semi-rigid connections can be accounted for by replacing the original moment of inertia of the girder I_g by a modified moment of inertia I_g'. With this in mind, a modified stiffness distribution factor G_r accounting for the effect of semi-rigid connections can be written as

$$G_r = \frac{\Sigma(EI_c/L_c)}{\Sigma(EI_g'/L_g)} \quad \text{(11)}$$

Note that as C approaches infinity, I_g' becomes I_g (Eq. 10) and so G_r reduces to G which corresponds to the case in which the girders are rigidly connected to the columns in the frame.

A similar approach can be followed to develop a modified moment of inertia of the girder I_g' for sway frames. For sway permitted frames, the assumption is that the rotation at the i-th end of the girder is equal to that of the j-th end when the column buckles. Thus by setting $\theta_j = \theta_i$ and $\theta_j' = \theta_i'$ and following the same procedure, the modified moment of inertia of the girder I_g' for the sway case can be shown to be

$$I_g' = I_g\left[\frac{1}{1 + 6EI_g/L_gC}\right] \quad \text{(12)}$$

The modified stiffness distribution factor can be written as in Eq. 11 and the alignment chart for the sway permitted case can be used to determine the K-factor.

For convenience, Eq. 11 can be rewritten as

$$G_r = \frac{\Sigma(EI_c/L_c)}{\Sigma(EI_g/L_g)(I_g'/I_g)} \quad \text{(13)}$$

The ratio I_g'/I_g for non-sway and sway cases are given by Eqs. 10 and 12 respectively. Plots of I_g'/I_g versus EI_g/L_gC for these two cases are shown in Fig. 16.

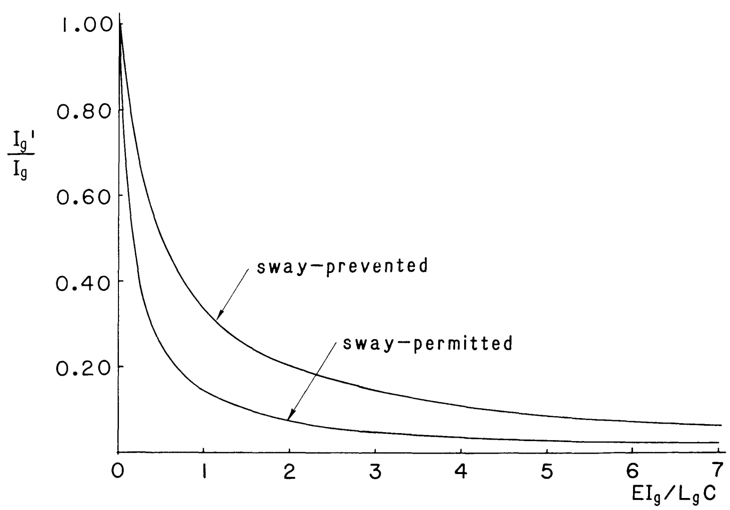

Fig. 16. I_g'/I_g vs. $EI_g/L_g C$

The stiffness distribution factor can be further modified to account for the inelastic action of the column,[27,28]

$$(G_r)_{inelastic} = G_r \frac{E_t}{E} \quad (14)$$

where

E_t = tangent modulus
E = modulus of elasticity

Reference 28, based on Ref. 27, recommends that the ratio E_t/E be taken as f_a/F'_e. There is a minor drawback in this approach. The shortcoming is that the factors of safety for f_a in the numerator and F'_e in the denominator are different. Although this inconsistency is not significant for practical purposes, a more accurate approach is to represent the ratio E_t/E by normalizing the CRC basis column curve by the Euler curve, i.e.

$$\frac{E_t}{E} = \frac{P_{CRC}}{P_E} = \begin{cases} \dfrac{1-0.25\lambda_c^2}{\lambda_c^{-2}} & \text{for } \lambda_c \le \sqrt{2} \\ 1 & \text{for } \lambda_c > \sqrt{2} \end{cases} \quad (15)$$

where

$$\lambda_c = \frac{KL_c}{r}\sqrt{\frac{F_y}{\pi^2 E}}$$

in which

L_c/r = slenderness ratio of the column
F_y = yield stress of the column
K = effective length factor

In most cases, K is not known in advance. However, by setting $f_a(=P/A)$ equal to F_a, KL_c/r can be estimated (see design example). Once KL_c/r is estimated, λ_c can be calculated and hence E_t/E can be evaluated using Eq. 15 or from Fig. 17. A similar approach to represent the ratio E_t/E was given by Smith.[30]

In order to account for the non-linear moment rotation (M-ϕ) behavior and subsequent deterioration of the rotational stiffness of the connection (Fig. 14), the conservative approach suggested by the author can be followed. In other words, the summation in the denominator of Eqs. 11 or 13 covers only the stiffnesses of the girders whose connections unload as the column buckles.

To sum up, the proposed modified inelastic stiffness distribution factor $(G_r)_{inelastic}$ is

$$(G_r)_{inelastic} = \frac{\Sigma(EI_c/L_c)}{\sum\limits_{unload}(EI_g/L_g)(I'_g/I_g)}\left(\frac{E_t}{E}\right) \quad (16)$$

The writers believe that Eq. 16 will provide a better alternative to the one proposed by the author. The ratio

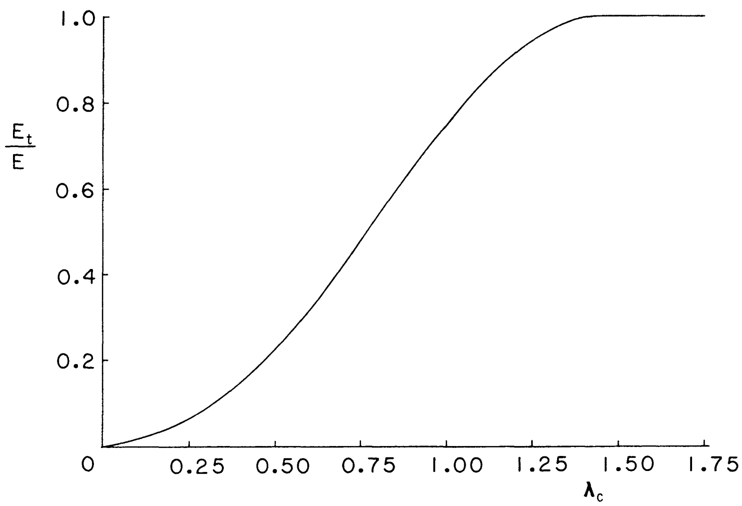

Fig. 17. E_t/E vs. λ_c

I'_g/I_g and E_t/E can easily be obtained from Figs. 16 and 17 once the connection and a trial column are picked.

To demonstrate the use of the above approach, the design example presented by the author is reworked below:

Interior column
axial load 200 kips
column length 14 ft
steel grade A36, F_y = 36 ksi
connections 2L with 3¾-in. A325 bolts
i.e. $C = 3.23 \times 10^4$ k-in./rad
Beam sizes W18×50 with 30 ft span on right side
W16×45 with 25 ft span on left side

Try W8×40 for column

$$I_x = 146 \text{ in.}^4,\ A = 11.7\text{in.}^2,\ r_x = 3.53 \text{ in.}$$

$$f_a = \frac{P}{A} = \frac{200}{11.7} = 17.1 \text{ ksi}$$

By setting $f_a = F_a$, KL_c/r can be estimated (Table 3-36, Appendix A, AISC Manual[11]) to be 63.5. Therefore,

$$\lambda_c = \frac{KL_c}{r}\sqrt{\frac{F_y}{\pi^2 E}} = (63.5)\sqrt{\frac{36}{\pi^2(290{,}000)}} = 0.712$$

$$(EI_c/L_c) = 2.52 \times 10^4$$

$$(EI_g/L_g) \text{ for W18×50} = 6.44 \times 10^4$$

$$(EI_g/L_g) \text{ for W16×45} = 5.66 \times 10^4$$

From Eq. 16, Figs. 16 and 17

$$\text{Top } (G_r)_{inelastic} = \frac{2\times 2.52\times 10^4}{(6.44\times 10^4)(0.2)}(0.443) = 1.73$$

Bottom $(G_r)_{inelastic}$

$$= \frac{2 \times 2.52 \times 10^4}{(8.566 \times 10^4)(0.222)} (0.443) = 1.78$$

From sway-prevented alignment chart,

$$K = 0.84$$

and

$$\frac{KL_c}{r} = \frac{0.84 \times 14 \times 12}{3.53} = 40(< 63.5)$$

Since the calculated KL/r (=40) is less than the estimated KL/r (=63.5), the trial column is **o.k.**

Note that the column is assumed to be braced in the weak direction so that bending about the strong axis in the plane of the restraining beams results upon application of the loads.

REFERENCES

29. *Driscoll, G.C.* Effective Length of Columns with Semi-Rigid Connections *AISC Engineering Journal, Vol. 13, No. 4, 4th Qtr., 1976, pp. 109-115.*
30. *Smith C.V.* On Inelastic Column Buckling *AISC Engineering Journal, Vol. 13, No. 3, 3rd Qtr., 1976, pp. 86-88.*

Closing discussion by **Reidar Bjorhovde**

The author is sincerely appreciative of the comments and constructive suggestions of Messrs. Gerstle, Lui and Chen. As is often the case when new design methods and procedures are proposed, improvements are always possible, and the discussers have pointed out some potential areas.

Gerstle's point of having to consider the moments that are developed in the design of the columns is well taken. It is agreed the end restraint procedure of the author will be of clear benefit to interior columns in a multi-story structure. The potential for similar levels of savings for exterior columns or for columns in one- or two-story frames needs further study. In fact, the procedure of the author recognizes the more limited effects of connection restraint insofar as exterior columns are concerned; specifically through the reduced value of the G_r term for such members. It is believed the results of the author and Gerstle will be quite close for exterior columns, and even more so as the size of the column increases. In other words, for larger columns, as noted by the author, the effect of end restraint will be reduced, but it must be borne in mind at the same time the moment that is thrown back into the column from the connection also will be of less significance.

Consequently, the author surmises that the design approach he has suggested will be of distinct benefit to most interior columns; that exterior columns of small to medium size will only gain a modest benefit; and that all columns of larger sizes will gain advantage. The procedure is not proposed for use with columns in one- or two-story structures; partly because of the uncertainty of the restraints that are afforded by the common connections in such structures, and partly because of the typical slenderness ratios that are used for these columns. Although the benefit of restraint is the highest for the most slender members, the moment that is thrown back to the column can be significant. It is recommended that a K-factor of 1.0 continue to be used for such members in braced (sway-prevented) structures.

The author is in agreement with Lui and Chen that their revised form for G_r may be more suitable. However, the results are very close to those of the author, as demonstrated by the example. It is believed both approaches will give satisfactory results in a practical design situation.

The use of the inelastic K-factor approach, as detailed by Yura[27] and Disque[28] has been recommended by the author and the discussers. The author agrees it is more appropriate to utilize E_t/E instead of $f_a F'_e$ in this procedure, due to the different levels of the factor of safety assigned to columns that buckle inelastically. Factoring the AISC column curve back to its origin (= the CRC curve) will remedy this inconsistency, but only for columns whose strength has been modeled by a tangent modulus approach. In other words, if a maximum strength model is used, where also the initial out-of-straightness of the member is incorporated, the approach suggested by the discussers will not suffice. Such a column strength model is used in the Canadian limit states design criteria,[29] as well as in the European column design recommendations.[30] In these cases, the modifying factor has to be P_{max}/P_E, since there is no convenient form that incorporates E-values. The maximum load ratios are easily developed, however, since they can be tabulated as the ratios of maximum strength from the appropriate column curve, divided by the elastic buckling strength.

The author does not agree with Chen that the procedure developed by the author, and improved by the discussers, should be applied to the design of columns in sway-permitted frames. Subsequent evaluations have shown the consequences for the beam moments will be significant, to the effect that as the column buckles, the beam moment will be magnified, and thus tends to overshadow the beneficial effects. However, this problem needs detailed study.

Reidar Bjorhovde is professor of Civil Engineering and Engineering Mechanics, University of Arizona, Tucson, Arizona.

REFERENCES

29. *Canadian Standards Association (CSA)* Steel Structures for Buildings — Limit States Design *CSA Standard No. S16.1–74, Rexdale, Ontario, Canada, December 1974.*
30. *European Convention for Constructional Steelworks (ECCS)* Manual for Steel Structures *ECCS, Liege, Belgium, April 1977.*