The Effective Length of Columns in Unbraced Frames

JOSEPH A. YURA

THE EFFECTIVE length concept for column design in unbraced frames has been incorporated in the AISC Specification since 1961. In simplified terms the concept is merely a method of mathematically reducing the problem of evaluating the critical stress for columns in structures to that of equivalent pinned-end braced columns. The Euler buckling stress for a column with both ends pinned and no sidesway,

$$F_e = \frac{\pi^2 E}{(l/r)^2} \tag{1}$$

can be used for all elastic column buckling problems by substituting an equivalent or effective column length Kl in place of the actual column length. The effective length factor K can be derived by performing a buckling analysis of the particular structure to determine the critical stress. The pinned-end column with an equivalent length which gives the same critical stress establishes the K-factor. The K-factor, then, is just a way of providing simple solutions to complex frame buckling problems.

The AISC Specification requires the determination of K-factors for columns in unbraced frames by some rational analysis, that is, a stability analysis. The elastic solutions that have been developed indicate that the buckling will always occur in a sidesway mode, as shown in Fig. 1a.[1] The buckling mode shown in Fig. 1b, in which no sidesway occurs, is a more stable condition, and it corresponds to the braced frame case. Bracing or some other shear resisting element must be present for the no-sidesway mode to govern. Bending moments in the columns due to beam gravity load do not significantly affect overall frame stability in the elastic range.[2]

Most engineers use the alignment chart shown in Fig. 2, which provides approximate elastic solutions (K-factors) in lieu of an actual stability analysis (which can be very complex). The solutions summarized in the chart are based on sidesway buckling of a simplified elastic structure. The alignment chart always indicates $K \geqslant 1.0$. K-factors between 2.0 and 3.0 and even larger values are not uncommon. The magnitude of these factors seems large to many engineers, especially since actual story heights were used in column design prior to 1961. Consequently, the effective length concept appears unreasonable to some designers.[3] However, as indicated earlier, the effective length concept is a rational method of adjusting a common formula for a variety of conditions. Therefore, if the factors appear unreasonable, the method of obtaining them, for example the alignment chart, should be examined more closely.

Much of the difficulty and misunderstanding of the effective length concept is due to the direct use of the alignment chart in situations which violate some of the basic assumptions in the derivation of the nomograph.

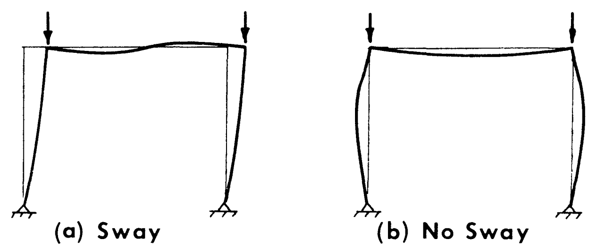

Fig. 1. Sway and no-sway buckling modes

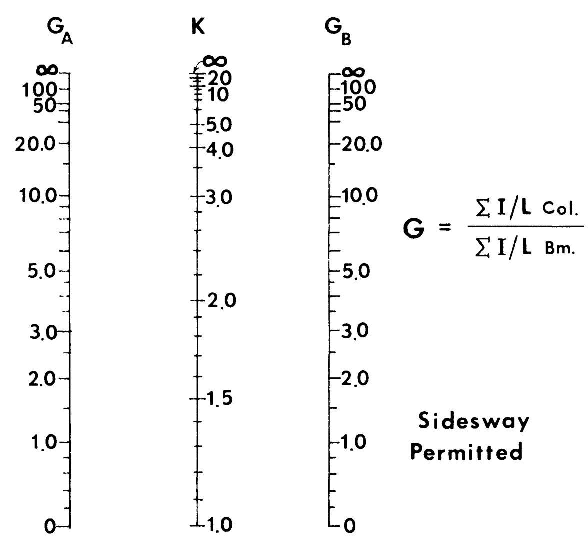

Fig. 2. Alignment chart for determining K-factors

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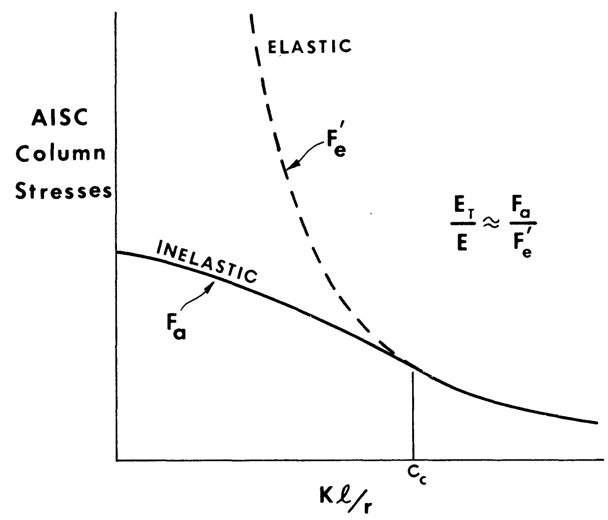

Fig. 3. AISC column stresses

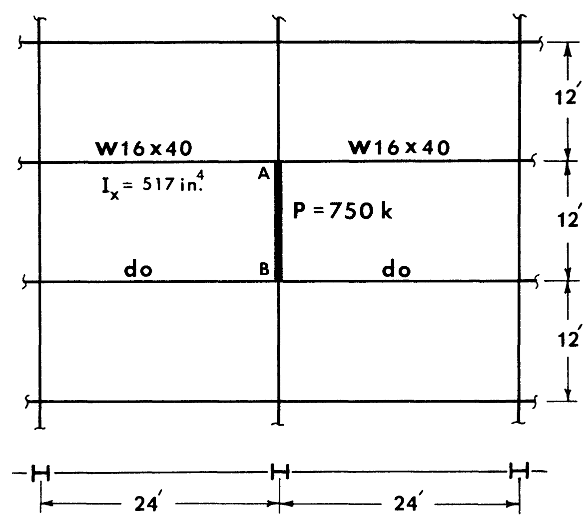

Fig. 4. Design Example 1

Two of the principal assumptions are:

1. Elastic action
2. All columns in a story buckle simultaneously

A variety of practical situations exists in which one or both of these assumptions are inaccurate and use of the alignment chart produces overly conservative designs. The development of design methods to handle such situations will now be presented.

INELASTIC COLUMNS

The AISC allowable column stress F_a is a function of Kl/r, as shown in Fig. 3. Because of residual stresses and initial out-of-straightness, inelastic action is assumed to begin at an average stress level of $0.5F_y$. The value of Kl/r corresponding to this stress level is called C_c, and it defines the assumed boundary between elastic and inelastic action. For A36 steel (F_y = 36 ksi), C_c = 126, which means that most columns in multistory structures will be in the inelastic range. The stiffness of a column in the elastic range is proportional to EI; however, the column stiffness in the inelastic range can be more accurately taken as proportional to a reduced stiffness E_TI, where E_T is the tangent modulus. The inelastic buckling stress is then

$$F_{cr} = \frac{\pi^2 E_T}{(l/r)^2} \qquad (2)$$

In the development of the alignment chart for column buckling in framed structures, the buckling strength was found to be related to

$$G = \frac{\text{Column Stiffness}}{\text{Beam Stiffness}} = \frac{\Sigma(EI/L)_{col}}{\Sigma(EI/L)_{beam}} \qquad (3)$$

In the elastic range, E cancels from Eq. (3), resulting in the familar expression for G given in Fig. 2. When the KL/r of a column is less than C_c, axial column behavior is inelastic. The column stiffness E_TI/L is less than the elastic value. Hence, an elastic beam offers more relative restraint to such a column. In the inelastic range of column behavior, the end restraint factor G may be defined by

$$G_{inelastic} = \frac{\Sigma(E_TI/L)_{col}}{\Sigma(EI/L)_{beam}} = \frac{E_T}{E} \cdot G_{elastic} \qquad (4)$$

The value of G normally used in the alignment chart is reduced by the factor E_T/E. If G is reduced, the effective length is also smaller. The effective length of columns in the inelastic range can be determined by the alignment chart if the reduced $G_{inelastic}$ is used. The only problem is to determine the factor E_T/E. This can be accomplished in a reasonably accurate manner by noting that for a given Kl/r,*

$$\frac{F_{cr(inelastic)}}{F_e} = \frac{E_t}{E} \approx \frac{F_a}{F'_e}$$

so Eq. (4) reduces to

$$G_{inelastic} = \frac{F_a}{F'_e} G_{elastic} \qquad (5)$$

In the elastic range, $F_a = F'_e$, so the nomograph procedures would be unchanged. Equation (5) has been developed by others in a slightly different form.[4,5] The use of Eq. (5) is rather simple, and its use will be illustrated in the following design example.

* *F_a and F_e' use different factors of safety, but this is ignored as a minor factor in the development.*

Design Example 1—Design column **AB** in the unbraced frame shown in Fig. 4 to support 750 kips, using A36 steel (F_y = 36 ksi). Only in-plane behavior will be considered for illustrative purposes. In the design, assume the column above and below are not significantly different in size.

Elastic Solution:

Try W14×150:

$I_x = 1790$ in.^{4}, $r_x = 6.37$ in., $A = 44.1$ in.^{2}

$$G_A = G_B = \frac{2\ (1790/12)}{2\ (517/24)} = 6.93$$

From the alignment chart, $K = 2.55$

$KL/r_x = 2.55\ (144)/6.37 = 58, \quad F_a = 17.62$ ksi

$P_{allow} = 17.62\ (44.1) = 776 > 750$ kips

Use W14×150.

Inelastic Solution:

Try W14×127:

$I_x = 1480$ in.^{4}, $r_x = 6.29$ in., $A = 37.3$ in.^{2}

The solution begins the same as the elastic solution.

$$\text{Elastic } G_A = G_B = \frac{2\ (1480/12)}{2\ (517/24)} = 5.72$$

From the alignment chart, $K = 2.35$

$KL/r_x = 2.35\ (144)/6.29 = 54 < C_c = 126$
∴ column is inelastic.

For $KL/r_x = 54$, $F_a = 17.99$, and $F'_e = 51.21$ ksi,
$G_{inelastic} = 5.72\ (17.99/51.21) = 2.01$

From the alignment chart, using $G = 2.01$, $K = 1.6$.

Thus K is already reduced significantly from its elastic value of 2.35. The designer can stop here after one cycle and use $K = 1.6$. Further reduction in K can be achieved by continuing to cycle the calculation until convergence is obtained. One more cycle will be illustrated:

Using the last value of K,

$KL/r_x = 1.6(144)/6.29 = 37;$
$F_a = 19.42, \quad F'_e = 109.1$

$$G_{inelastic} = \frac{19.42}{109.2}\ (G_{elastic} = 5.72) = 1.02$$

From the alignment chart, $K = 1.3$

A few more cycles show that $K \approx 1.0$.

$KL/r_x = 1.0\ (144)/6.29 = 23, \quad F_a = 20.41$

$P_{allow} = 37.3\ (20.41) = 762 > 750$

Use W14×127.

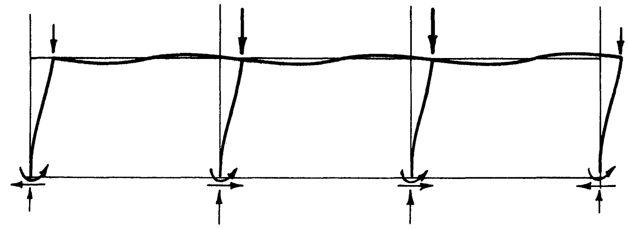

Fig. 5. Column bracing in unbraced frames

The design example shows that the inelastic approach can produce significant reductions in the effective length factor. When the elastic KL/r is reasonably low (about 50 or less), the actual K will usually converge to 1.0, although no specific rule has yet been established. This observation indicates that columns in multistory frames can often be designed on the basis of $K = 1.0$; that is, the actual story height. Studies by Lu[5] on inelastic frame buckling also indicate that, in the low slenderness range, the sidesway and no sidesway buckling modes shown in Fig. 1 converge in contrast to the elastic solutions, in which the sidesway mode ($K > 1.0$) always governs.

In the moderate slenderness range, the inelastic solution will converge in about two cycles. When KL/r is greater than C_c the elastic solution is valid and cannot be reduced.

BUCKLING STRENGTH OF A STORY

Designs based on the alignment chart are reasonably accurate only when all the individual columns in a story buckle simultaneously under their individual proportionate share of the total gravity load. The columns can not brace each other in this situation — their total strength is required to support their own gravity loads, leaving no reserve which might be counted upon to provide a bracing force for other columns. There are situations in which the individual columns have excessive buckling strengths. Such a condition is shown in Fig. 5. If the two exterior columns contain axial loads such that the buckling load of these columns is not reached when the interior columns reach their independent buckling loads, the system will not buckle. This may occur when different loading conditions govern the design of various columns in a story. Shear resistance will be developed in the exterior columns which counteracts the sidesway tendency. (If all columns want to buckle simultaneously, there will be no shear resistance available. In Fig. 5, the exterior columns "brace" the interior columns. Only when the gravity loading is increased enough to offset the stabilizing effect of the lightly loaded exterior columns will buckling occur. The critical load for the interior columns is increased and their effective length is decreased. The stabilizing effect can be such that the effective length of some of the columns could be reduced to 1.0, even though there is no apparent bracing system.

It is safe to treat separately each column to which beams are rigidly attached and to use the alignment chart to get the individual strengths. However, in some instances this usual approach may be unduly conservative. In the following section, a simple design approach will be developed which considers the potential bracing capacity of columns in a story. The approach will be illustrated by two practical design examples for which exact solutions are difficult to achieve.

Sidesway buckling is a total story phenomenon. A single individual column cannot fail by sidesway without all the columns in the same story also buckling in a sway mode. On the other hand, buckling in a non-sway mode is an individual phenomenon. Each column's non-sway buckling load is reasonably independent of the buckling load of the other columns. A quantitative description of frame buckling is presented in Fig. 6. In Fig. 6a, the frame is unbraced. The column sizes were chosen so that both columns buckle at the individual loads shown and in each case the effective length is 2.0. When sidesway occurs, the base moments $P\Delta$ are produced. For the given situation, the total load on the frame is 600 and the $P\Delta$ moments total 600Δ. If the frame is braced as shown in Fig. 6b, so that $K = 1.0$, each column can carry four times the load that can be supported by the unbraced frame. In the braced frame, the maximum load the left column can carry is 400, independent of the load on the right column.

Suppose that the frame is unbraced as in Fig. 6a, but the column on the right supports a load of 300, a reduction of 200 from its individual unbraced strength. The loading configuration is shown in Fig. 6c. This column will not sidesway until a moment of 500Δ is reached at the base, so the column has the capability to sustain an additional moment of 200Δ from another source. In other words, the right column has a reserve of strength which can be utilized to provide a bracing force to prevent sidesway of the left-hand column as long as the brace requirement does not exceed $200\Delta/l$. Since the column on the left is now braced and sidesway will not occur until the base moment reaches 100Δ, it can be designed with a K-factor of less than 2.0; therefore, it can support an additional load of 200 (total load = 300). This is 200 greater than the individual unbraced column, but less than the capacity as a fully braced, non-sidesway column. Note that the total frame load at buckling is still 600, the same as in Fig. 6a; however, the load distribution on the individual columns is different. To summarize, the shear capacity of a column can be replaced (approximately) by an equivalent axial load. The conservativeness of this approach for the structure shown is illustrated in Fig. 7.

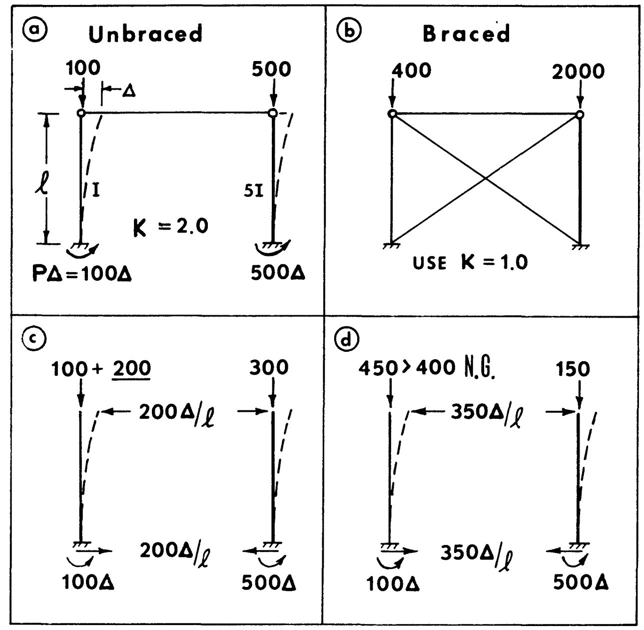

Fig. 6. Quantitative description of frame instability

Taking Y as an axial load produces more moment area than using $Y\Delta/l$ as an assumed bracing force.

In general, the total gravity loads which produce sidesway can be distributed among the columns in a story in any manner. *Sidesway will not occur until the total frame load on a story reaches the sum of the* **potential** *individual column loads for the* **unbraced** *frame.* There is one limitation; the maximum load an individual column can carry is limited to the load permitted on that column for the braced case, $K = 1.0$. Such a situation is illustrated in Fig. 6d. Even though the column on the right can support an additional overturning moment of 350Δ from a source other than its own load, the maximum load on the left column cannot exceed 400. In this case, the left column is fully braced by the one on the right. The left column will fail in a non-sway mode.

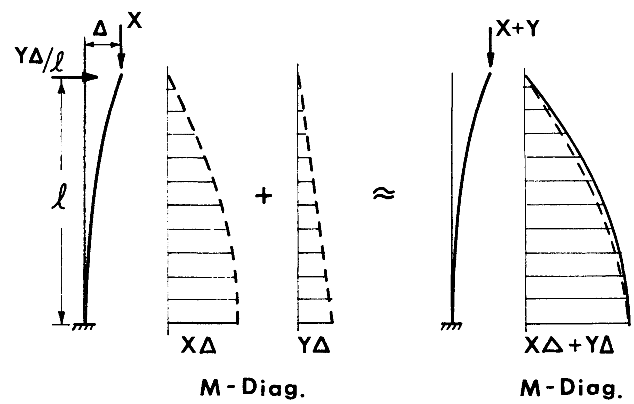

Fig. 7. Replacement of shear by equivalent axial load

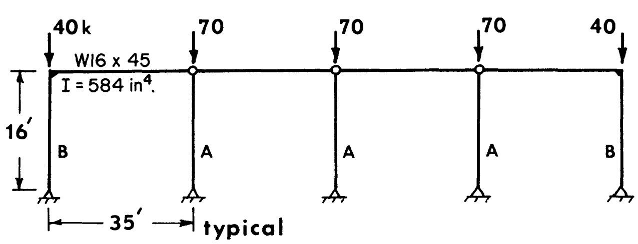

Fig. 8. Design Example 2

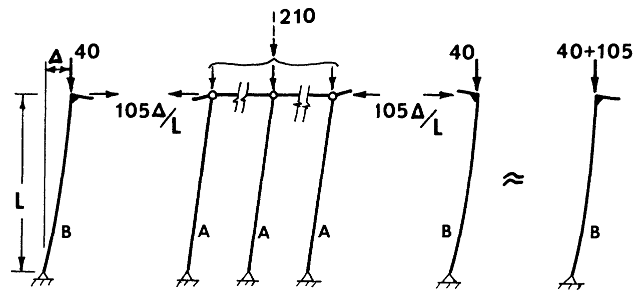

Fig. 9. Bracing force as equivalent column load

Although the concept has been demonstrated on the simple structure shown in Fig. 6, Salem[6] has shown theoretically that this concept holds regardless of the type of framing, ratio of member sizes, etc. Sidesway is a total story characteristic, not an individual column phenomenon; the advantage of this fact will be illustrated in the two examples which follow.

Design Example 2—Design the columns in the structure shown in Fig. 8. Rigid connections are used only at the exterior columns; simple connections are used at columns **A**. Each column is braced top and bottom out of the plane of the frame. Sidesway is permitted in-plane but not out-of-plane. $F_y = 36$ ksi.

Columns A:

Out-of-plane: $K = 1.0$

In-plane: direct use of the alignment chart for a column with both ends pinned and sidesway not prevented gives $K = \infty$, i.e., the column is unstable. However, the exterior columns can be designed to stabilize the system, so use $K = 1.0$.

From the AISC Manual,
$KL = 1.0\ (16) = 16$ ft

Use W8×24 ($P_{allow} = 73 > 70$ kips)

Columns B:

Out-of-plane (Y-Y axis): $K = 1.0$, $P = 40$ kips

In-plane (X-X axis): columns **B** stabilize the structure for sidesway. In addition to its own load, each column **B** must support an additional $P\Delta$ moment = 105Δ. As shown in Fig. 9, this is equivalent to an additional axial load on the exterior column. Consequently, the exterior column must be able to support a fictitious axial load of $40 + 105 = 145$ kips in-plane only. Determine K from the alignment chart. There is no change out-of-plane, since bracing against sidesway is provided at individual columns.

Try W12×31:

Y-Y axis: $P = 40$ kips, $KL = 16$ ft,
$P_{allow} = 87 > 40$ kips

X-X axis: $P = 145$ kips, $I_{col} = 239$ in.[4], $K = ?$

$$G_{top} = \frac{\Sigma\,(I/L)_{col}}{\Sigma\,(I/L)_{beam}} = \frac{239/16}{(584/35) \times 2.0} = 1.78$$

Note: Beam length was increased by a factor of 2.0 because far end of beam is pinned.[1]

$G_{bot} = 10$ (pinned base)

From the alignment chart, $K = 2.0$.

From the AISC Manual,

$$\frac{KL}{r_x} = \frac{2.0\ (16 \times 12)}{5.12} = 75$$

$F_a = 15.90$ ksi

$P_{allow} = 145.2 > 145$ kips

Use W12×31.

The example above has been considered elsewhere,[7] but that solution required the use of charts not available in the AISC Manual. In addition, the approach above provides a better insight to the required size of the column.

Design Example 3—A portion of a large unbraced one-story industrial plant is shown in Fig. 10. The deep roof trusses have infinite rigidity compared to the columns,

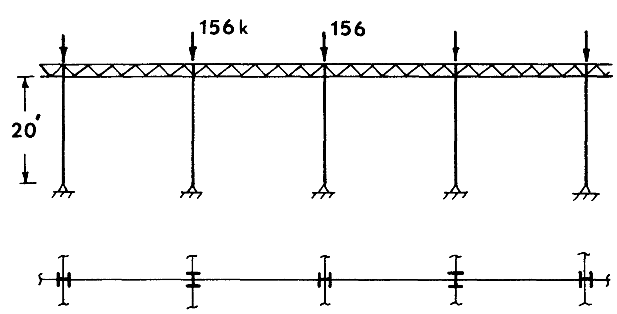

Fig. 10. Design Example 3

and the trusses frame in two directions; in-plane and out-of-plane. Adjacent columns have their strong axes turned 90° in order to equalize sway stiffness in the two main directions of the building. Design the columns using A36 steel.

$G_{top} = 0$, $G_{bot} = 10$ (pinned base). From the alignment chart, $K = 1.65$. $KL = 1.65\ (20) = 33$ ft, both axes, both building directions. Two possible solutions will be considered.

Solution A:

The standard solution treats each column separately. $P = 156$ kips, $KL = 33$ ft, weak axis governs.

From Column Tables, AISC Manual:

Use W12×65

Solution B:

The alternate design method takes advantage of the fact that two adjacent columns do not attempt to sway in the same direction simultaneously. The column with the stronger axis in the plane of the frame braces the adjacent column with its weaker axis in the plane.

Try W10×48: Check the strength of two adjacent columns, which must support a total load of 312 kips. Use the column load tables in the AISC Manual.

X-X axis: $KL = 33$ ft, $P_{allow} = 203$ kips
Y-Y axis: $KL = 33$ ft, $P_{allow} = 88$ kips
$P_{total} = 291 \text{ kips} < 312 \text{ kips}$
N. G.

Try W12×53:
X-X axis: $KL = 33$ ft, $P_{allow} = 246$ kips
Y-Y axis: $KL = 33$ ft, $P_{allow} = 91$ kips
$P_{total} = 337 \text{ kips} > 312 \text{ kips}$
O. K.

Use W12×53

Note that P_{allow} for sidesway buckling about the Y-Y axis ($Kl = 33$ ft) is less than the applied load, but 91 kips is the load which can be supported without any bracing. Since the applied load on the X-X column is less than the individual critical load, it can provide some bracing to the Y-Y column and hence increase the capacity above 91 kips. The X-X column can provide bracing to permit a load increase of (246 − 156) = 90 kips on the adjacent column. So the approximate capacity of the Y-Y column is (91 + 90) = 181 kips, which is greater than the applied load.

The alternate approach permits a lighter column, because consideration is given to the bracing effect of less critically loaded columns. The design is the same as that obtained by Zweig[8] using a more exact and complex procedure.

SUMMARY

The alignment chart gives valid sidesway buckling solutions if the columns are in the elastic range ($Kl/r > C_c$) and all columns in a story reach their individual buckling loads simultaneously. For columns that do not satisfy these two conditions, the chart is generally overly conservative or not applicable at all. The design methods presented herein handle these situations by adjusting common procedures: the Column Load Tables in the AISC Manual and the alignment chart can still be utilized.

The concepts developed are applicable to a wide range of problems. For example, using the concept of some columns bracing others in the same story of an unbraced frame, it is possible to make the column sizes in a story reasonably uniform, even though individual column loads might vary substantially.

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Discussion

The Effective Length of Columns in Unbraced Frames

Paper presented by JOSEPH A. YURA (April 1971 issue)

Discussion by **JOSEPH A. YURA**

IN PREVIOUS discussion to this April 1971 paper, questions were raised dealing with the applicability of the inelastic effective length factor in the interaction equation for beam-columns, Formula (1.6-1a) of the AISC Specification. It was recommended that the inelastic effective length be used in evaluating F_a, but that the elastic effective length be used in calculating F_e'. No mathematical data were provided to support this recommendation, so the purpose of this discussion is to present such data.

Formula (1.6-1a) is based on the following interaction equation:

$$\frac{P}{P_{cr}} + \frac{C_m M}{\left(1 - \frac{P}{P_e}\right) M_p} = 1.0 \qquad (1)$$

for beam columns with lateral bracing (no out-of-plane buckling). Equation (1) cannot be derived directly from material strength and stress conditions; rather, its applicability has been verified by comparisons with reasonably exact theoretical solutions and test results. Based on a theoretical study of unbraced single-story frames,[1] it was established that Eq. (1) produced safe, reasonable results if the effective length factor for the sidesway permitted case was used in calculating P_{cr} and P_e, and C_m was chosen as 0.85. The comparisons are reproduced in Figs. 1a and 1b, which show that the interaction equation generally underestimates the actual strength of the frame. In Fig. 1, $P_y = AF_y$ and Q_p is the lateral load producing simple plastic failure neglecting axial loads.

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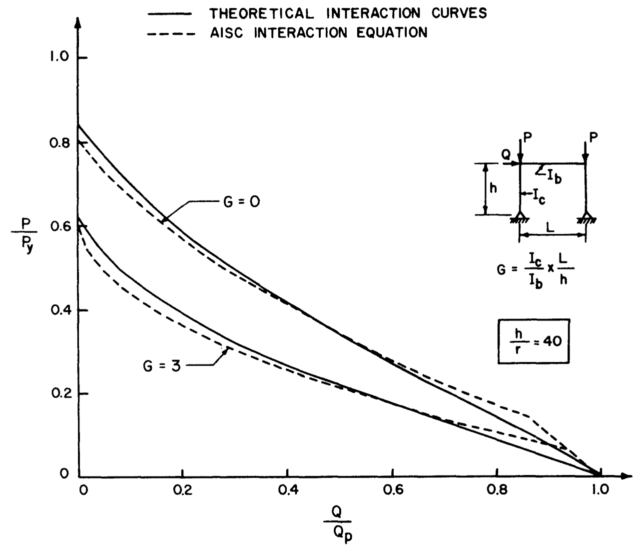

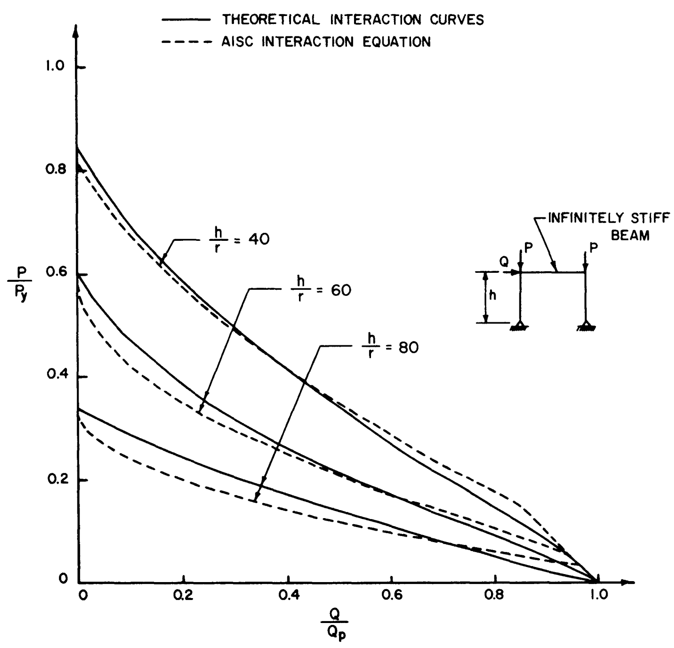

Fig. 1. Frame behavior in the elastic range

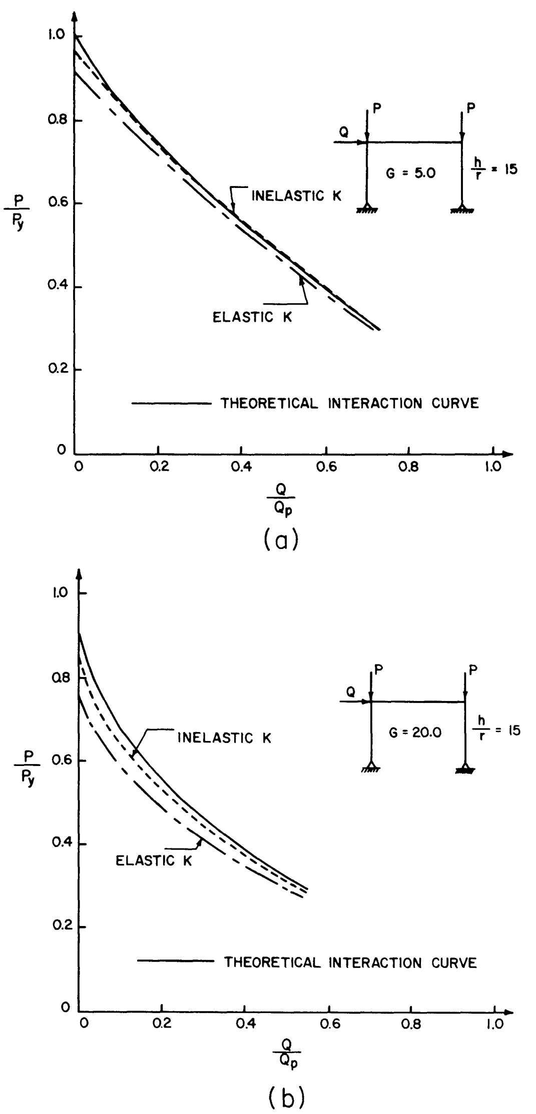

Fig. 2. Frame behavior with columns in the inelastic range

Unfortunately, the cases shown in Fig. 1 use either elastic effective length factors or a value of 2.0 corresponding to the infinitely stiff beam case. Neither of these situations provide any data related to the use of inelastic effective length factors. Consequently the frames shown in Fig. 2 have been solved in which the columns are in the inelastic range. The two cases shown in Fig. 2 were chosen to provide a wide range of G values and a significant difference between the elastic and inelastic effective length factors. For $G = 5.0$, $K_{elastic} = 3.4$ and $K_{inelastic} = 2.3$; for $G = 20.0$, $K_{elastic} = 6.0$ and $K_{inelastic} = 4.5$. The comparisons of Eq. (1) using both elastic and inelastic effective length factors in the first term with the theoretical solution based on the Lehigh Subassemblage Design Charts[2] show that the solution with $K_{inelastic}$ produces conservative results which are in better agreement with the theory.

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